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Fuzzy MultiDirected Sets, Neutrosophic MultiDirected Sets, Plithogenic MultiDirected Sets, Soft MultiDirected Sets, and Hypersoft MultiDirected Sets

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Abstract

A fuzzy set generalizes classical sets by assigning each element a membership value in the interval $[0, 1]$, thereby capturing partial or uncertain membership. A *Neutrosophic Set* extends fuzzy sets by introducing three independent membership degrees: truth, indeterminacy, and falsity, each ranging within $[0, 1]$. A *Plithogenic Set* further generalizes classical and fuzzy sets by incorporating attributes, their possible values, and a measure of contradiction. Other notable frameworks for handling uncertainty include the *Soft Set* and *Hypersoft Set*, which provide flexible parameterized approaches. Additionally, concepts such as the *Directed Set* and *MultiDirected Set*, which introduce directionality to sets, are also well-established. This paper explores and analyzes the structures of Fuzzy Directed Sets, Neutrosophic Directed Sets, Plithogenic Directed Sets, Fuzzy MultiDirected Sets, Neutrosophic MultiDirected Sets, Plithogenic MultiDirected Sets, Soft Directed Sets, Hypersoft Directed Sets, Soft MultiDirected Sets, and Hypersoft MultiDirected Sets.

Keywords: Plithogenic set, Fuzzy Set, Neutrosophic set, Directed Set, MultiDirected Set

1 | Preliminaries and Definitions

This section outlines the key concepts and definitions necessary for the discussions in this paper. The study builds upon classical set theory and extends its principles into more advanced theoretical frameworks. For readers seeking a deeper understanding of fundamental set theory, recommended resources include [1]. This paper deals exclusively with finite sets.



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1.1 | Uncertain Set

To model uncertainty, vagueness, and imprecision in decision-making, various extensions of classical set theory have been proposed. Among these, *Fuzzy Sets*, introduced by Zadeh in his pioneering works [2, 3], allow each element to have a membership degree in the range $[0, 1]$, enabling the representation of partial belonging. A further refinement, the *Intuitionistic Fuzzy Set*, extensively studied by Atanassov and others [4], extends fuzzy sets by incorporating an additional degree of non-membership, thereby capturing a broader range of uncertainty.

Neutrosophic Sets, introduced by Smarandache, offer a flexible and comprehensive framework for handling indeterminacy in uncertain environments [5, 6, 7, 8, 9]. These sets define three independent membership functions—truth, indeterminacy, and falsity—each taking values in $[0, 1]$. This approach has led to further generalizations, such as the *Hyperneutrosophic Set*, which expands the neutrosophic framework to higher-order structures [10, 11, 12].

More recently, the concept of *Plithogenic Sets*, also introduced by Smarandache, has gained attention for its capability to model multi-dimensional uncertainty and contradictions in complex systems [13, 14, 15, 16]. These sets extend classical and fuzzy set concepts by incorporating attribute-based membership functions and a contradiction degree, allowing for a more nuanced representation of uncertainty.

Definition 1 (Fuzzy set). [3] A *fuzzy set* τ in a non-empty universe Y is a mapping $\tau : Y \rightarrow [0, 1]$. A *fuzzy relation* on Y is a fuzzy subset δ in $Y \times Y$. If τ is a fuzzy set in Y and δ is a fuzzy relation on Y , then δ is called a *fuzzy relation on τ* if

$$\delta(y, z) \leq \min\{\tau(y), \tau(z)\} \quad \text{for all } y, z \in Y.$$

Definition 2 (Neutrosophic Set). [5, 6] Let X be a non-empty set. A *Neutrosophic Set (NS)* A on X is characterized by three membership functions:

$$T_A : X \rightarrow [0, 1], \quad I_A : X \rightarrow [0, 1], \quad F_A : X \rightarrow [0, 1],$$

where for each $x \in X$, the values $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the degrees of truth, indeterminacy, and falsity, respectively. These values satisfy the following condition:

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3.$$

Definition 3 (Plithogenic Set). [15, 17] Let S be a universal set, and $P \subseteq S$. A *Plithogenic Set PS* is defined as:

$$PS = (P, v, Pv, pdf, pCF)$$

where:

- v is an attribute.
- Pv is the range of possible values for the attribute v .
- $pdf : P \times Pv \rightarrow [0, 1]^s$ is the *Degree of Appurtenance Function (DAF)*.
- $pCF : Pv \times Pv \rightarrow [0, 1]^t$ is the *Degree of Contradiction Function (DCF)*.

These functions satisfy the following axioms for all $a, b \in Pv$:

(1) *Reflexivity of Contradiction Function:*

$$pCF(a, a) = 0$$

(2) *Symmetry of Contradiction Function:*

$$pCF(a, b) = pCF(b, a)$$

1.2 | Hypersoft Set

A Soft Set provides a simplified approach to parameterized decision modeling by mapping attributes, or parameters, to subsets of a universal set, thereby addressing uncertainty in a straightforward manner [18]. Expanding on this idea, a Hypersoft Set facilitates enhanced multi-attribute decision analysis by mapping combinations of multiple attributes to subsets of a universal set [19, 20, 21, 22].

Definition 4 (Soft Set). [18, 23] Let U be a universal set and A be a set of attributes. A soft set over U is a pair $(\mathcal{F}, S)$, where $S \subseteq A$ and $\mathcal{F} : S \rightarrow \mathcal{P}(U)$. Here, $\mathcal{P}(U)$ denotes the power set of U . Mathematically, a soft set is represented as:

$$(\mathcal{F}, S) = \{(\alpha, \mathcal{F}(\alpha)) \mid \alpha \in S, \mathcal{F}(\alpha) \in \mathcal{P}(U)\}.$$

Each $\alpha \in S$ is called a parameter, and $\mathcal{F}(\alpha)$ is the set of elements in U associated with α .

Definition 5 (Hypersoft Set). [24] Let U be a universal set, and let $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_m$ be attribute domains. Define $\mathcal{C} = \mathcal{A}_1 \times \mathcal{A}_2 \times \dots \times \mathcal{A}_m$, the Cartesian product of these domains. A hypersoft set over U is a pair $(G, \mathcal{C})$, where $G : \mathcal{C} \rightarrow \mathcal{P}(U)$. The hypersoft set is expressed as:

$$(G, \mathcal{C}) = \{(\gamma, G(\gamma)) \mid \gamma \in \mathcal{C}, G(\gamma) \in \mathcal{P}(U)\}.$$

For an m -tuple $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_m) \in \mathcal{C}$, where $\gamma_i \in \mathcal{A}_i$ for $i = 1, 2, \dots, m$, $G(\gamma)$ represents the subset of U corresponding to the combination of attribute values $\gamma_1, \gamma_2, \dots, \gamma_m$.

1.3 | Directed Set and MultiDirected Set

The definitions of directed set and multi-directed set are provided below.

Definition 6 (Directed set). (cf.[25, 26, 27]) A *directed set* is a pair $(D, \prec)$ where:

- D is a nonempty set.
- $\prec$ is a binary relation on D satisfying:
 - *Acyclicity*: There do not exist elements $x_1, x_2, \dots, x_n \in D$ such that

$$x_1 \prec x_2 \prec \dots \prec x_n \prec x_1.$$
 - *Transitivity*: If $x \prec y$ and $y \prec z$, then $x \prec z$.

The elements of D are called *nodes*, and the pairs $(x, y) \in \prec$ are called *directed relations*.

Example 7 (Directed Set: Project Task Dependencies). Consider a small software project with four tasks:

$$D = \{\text{Design, Implement, Test, Deploy}\}.$$

We define a strict precedence relation $\prec$ on D as follows:

$$\prec = \{(\text{Design, Implement}), (\text{Implement, Test}), (\text{Test, Deploy}), (\text{Design, Test}), (\text{Design, Deploy}), (\text{Implement, Deploy})\}.$$

- *Acyclicity*: There is no sequence of tasks that leads back to itself (e.g. no cycle $\text{Design} \prec \text{Implement} \prec \text{Test} \prec \text{Design}$).
- *Transitivity*: Since $\text{Design} \prec \text{Implement}$ and $\text{Implement} \prec \text{Test}$, we include $\text{Design} \prec \text{Test}$; similarly for all implied pairs.

Thus $(D, \prec)$ is a directed set modeling the required order of project tasks.

Definition 8 (Multi-directed set). [25, 28] A *multi-directed set* generalizes the notion of a directed set by allowing multiple directed relations between the same pair of elements. It is a quadruple (M, R, i, t) where:

- M is a nonempty set of *nodes*.
- R is a nonempty set of *relations*.
- $i : R \rightarrow M$ is the *initial element function*, assigning each relation an initial node.

- $t : R \rightarrow M$ is the *terminal element function*, assigning each relation a terminal node.

A *chain* in a multi-directed set is a sequence of relations $r_1, r_2, \dots, r_k$ such that

$$t(r_i) = i(r_{i+1}) \quad \text{for all } 1 \leq i < k.$$

Example 9 (Multi-Directed Set: Airline Flight Network). Let

$$M = \{\text{NYC, LAX, CHI, MIA}\}$$

be a set of airport codes, and let R be the set of individual flights (relations) with possibly multiple flights between the same cities:

$$\begin{aligned} R &= \{f_1, f_2, f_3, f_4, f_5, f_6\}, \\ i(f_1) &= \text{NYC}, \quad t(f_1) = \text{LAX}, \\ i(f_2) &= \text{NYC}, \quad t(f_2) = \text{LAX}, \\ i(f_3) &= \text{LAX}, \quad t(f_3) = \text{CHI}, \\ i(f_4) &= \text{LAX}, \quad t(f_4) = \text{CHI}, \\ i(f_5) &= \text{CHI}, \quad t(f_5) = \text{MIA}, \\ i(f_6) &= \text{NYC}, \quad t(f_6) = \text{MIA}. \end{aligned}$$

Here:

- There are two distinct flights f_1, f_2 from NYC to LAX, and two f_3, f_4 from LAX to CHI.
- A chain of flights f_1, f_3, f_5 corresponds to a passenger traveling $\text{NYC} \rightarrow \text{LAX} \rightarrow \text{CHI} \rightarrow \text{MIA}$.

Hence (M, R, i, t) is a multi-directed set representing an airline's flight network with multiple scheduled connections between hubs.

2 | Results of This Paper

This section presents the main findings discussed in this paper.

2.1 | Fuzzy Directed Set

The definition of the Fuzzy Directed Set is provided below.

Definition 10 (Fuzzy Directed Set). A *fuzzy directed set* is a pair (D, F) where:

- D is a nonempty set.
- $F : D \times D \rightarrow [0, 1]$ is a fuzzy binary relation satisfying:

(1) *Fuzzy Transitivity*: For all $x, y, z \in D$,

$$F(x, z) \geq \min\{F(x, y), F(y, z)\}.$$

(2) *Fuzzy Acyclicity*: For any finite cycle $x_1, x_2, \dots, x_n \in D$ (with $n \geq 2$),

$$\min\{F(x_1, x_2), F(x_2, x_3), \dots, F(x_n, x_1)\} = 0.$$

Example 11. Let $D = \{a, b, c\}$ and define $F : D \times D \rightarrow [0, 1]$ by:

$$F(a, b) = 0.8, \quad F(b, c) = 0.7, \quad F(a, c) = 0.7,$$

and $F(x, y) = 0$ for all other pairs. Then (D, F) is a fuzzy directed set.

Theorem 12 (Nonzero Composition in Fuzzy Directed Sets). *In a fuzzy directed set (D, F) , if $F(x, y) > 0$ and $F(y, z) > 0$ for some $x, y, z \in D$, then*

$$F(x, z) \geq \min\{F(x, y), F(y, z)\} > 0.$$

Proof: This follows directly from the fuzzy transitivity property. □

Theorem 13 (Path-Composition Induction). *Let (D, F) be a fuzzy directed set. For any finite sequence $x_0, x_1, \dots, x_n$ in D , define*

$$\delta(x_0 \rightarrow x_n) = \min_{1 \leq i \leq n} F(x_{i-1}, x_i).$$

Then

$$F(x_0, x_n) \geq \delta(x_0 \rightarrow x_n),$$

and in particular if each $F(x_{i-1}, x_i) > 0$ then $F(x_0, x_n) > 0$.

Proof: We prove by induction on n . For $n = 1$ the claim is trivial. Suppose it holds for sequences of length n . Given $x_0, \dots, x_{n+1}$,

$$\delta(x_0 \rightarrow x_{n+1}) = \min\{\delta(x_0 \rightarrow x_n), F(x_n, x_{n+1})\}$$

and by the inductive hypothesis,

$$F(x_0, x_n) \geq \delta(x_0 \rightarrow x_n),$$

so by fuzzy transitivity,

$$F(x_0, x_{n+1}) \geq \min\{F(x_0, x_n), F(x_n, x_{n+1})\} \geq \min\{\delta(x_0 \rightarrow x_n), F(x_n, x_{n+1})\} = \delta(x_0 \rightarrow x_{n+1}). \quad \square$$

Theorem 14 (Antisymmetry from Acyclicity). *In a fuzzy directed set (D, F) , for any two distinct $x, y \in D$,*

$$\min\{F(x, y), F(y, x)\} = 0.$$

Proof: If both $F(x, y)$ and $F(y, x)$ were strictly positive, then by fuzzy transitivity we would have

$$F(x, x) \geq \min\{F(x, y), F(y, x)\} > 0,$$

yielding a cycle (x, y, x) of positive minimum weight, contradicting fuzzy acyclicity. Hence one of them must be zero. $\square$

Theorem 15 (Intersection Closure). *Let $\{(D, F_i)\}_{i \in I}$ be a family of fuzzy directed sets on the same D . Define*

$$F(x, y) = \min_{i \in I} F_i(x, y).$$

Then (D, F) is also a fuzzy directed set.

Proof: Fuzzy transitivity:

$$F(x, z) = \min_i F_i(x, z) \geq \min_i \min\{F_i(x, y), F_i(y, z)\} = \min\{\min_i F_i(x, y), \min_i F_i(y, z)\} = \min\{F(x, y), F(y, z)\}.$$

Fuzzy acyclicity: For any cycle $x_1, \dots, x_n$,

$$\min_j F(x_j, x_{j+1}) = \min_j \min_i F_i(x_j, x_{j+1}) = \min_i \min_j F_i(x_j, x_{j+1}) = 0,$$

since each F_i has at least one zero-weight edge on the cycle. $\square$

Theorem 16 (Scalar Scaling). *Let (D, F) be a fuzzy directed set and let $\lambda \in [0, 1]$. Define*

$$F_\lambda(x, y) = \lambda F(x, y).$$

Then (D, F_λ) is also a fuzzy directed set.

Proof: Fuzzy transitivity:

$$F_\lambda(x, z) = \lambda F(x, z) \geq \lambda \min\{F(x, y), F(y, z)\} = \min\{\lambda F(x, y), \lambda F(y, z)\} = \min\{F_\lambda(x, y), F_\lambda(y, z)\}.$$

Fuzzy acyclicity: For any cycle $x_1, \dots, x_n$,

$$\min_j F_\lambda(x_j, x_{j+1}) = \lambda \min_j F(x_j, x_{j+1}) = \lambda \cdot 0 = 0.$$

$\square$

2.2 | Neutrosophic Directed Set

The definition of the Neutrosophic Directed Set is provided below.

Definition 17 (Neutrosophic Directed Set). A *neutrosophic directed set* is a pair (D, R) where:

- D is a nonempty set.
- $R : D \times D \rightarrow [0, 1]^3$ is a neutrosophic binary relation. For each $(x, y) \in D \times D$,

$$R(x, y) = (T_R(x, y), I_R(x, y), F_R(x, y)),$$

with

$$0 \leq T_R(x, y) + I_R(x, y) + F_R(x, y) \leq 3.$$

The relation R is required to satisfy:

- (1) *Neutrosophic Transitivity*: For all $x, y, z \in D$,

$$T_R(x, z) \geq \min\{T_R(x, y), T_R(y, z)\}.$$

- (2) *Neutrosophic Acyclicity*: For any finite cycle $x_1, x_2, \dots, x_n$ in D (with $n \geq 2$),

$$\min\{T_R(x_1, x_2), T_R(x_2, x_3), \dots, T_R(x_n, x_1)\} = 0.$$

Example 18. Let $D = \{a, b, c\}$ and define $R : D \times D \rightarrow [0, 1]^3$ by:

$$R(a, b) = (0.8, 0.1, 0.1), \quad R(b, c) = (0.7, 0.2, 0.1), \quad R(a, c) = (0.7, 0.15, 0.15),$$

and $R(x, y) = (0, 0, 0)$ for all other pairs. Then (D, R) is a neutrosophic directed set.

Theorem 19 (Neutrosophic Transitivity Implication). *In a neutrosophic directed set (D, R) , if $T_R(x, y) > 0$ and $T_R(y, z) > 0$ for some $x, y, z \in D$, then*

$$T_R(x, z) \geq \min\{T_R(x, y), T_R(y, z)\} > 0.$$

Proof: This follows immediately from the neutrosophic transitivity condition. $\square$

Theorem 20. *Let (D, F) be a Fuzzy Directed Set, where $F : D \times D \rightarrow [0, 1]$ satisfies the fuzzy transitivity condition:*

$$F(x, z) \geq \min\{F(x, y), F(y, z)\} \quad \text{for all } x, y, z \in D.$$

Then, define a neutrosophic relation $R : D \times D \rightarrow [0, 1]^3$ by

$$R(x, y) = (F(x, y), 0, 1 - F(x, y)).$$

The pair (D, R) forms a Neutrosophic Directed Set. Thus, every Fuzzy Directed Set can be embedded into a Neutrosophic Directed Set.

Proof: For each $x, y \in D$, set

$$T_R(x, y) = F(x, y), \quad I_R(x, y) = 0, \quad F_R(x, y) = 1 - F(x, y).$$

Since $F(x, y) \in [0, 1]$, it follows that

$$0 \leq T_R(x, y) + I_R(x, y) + F_R(x, y) = F(x, y) + 0 + (1 - F(x, y)) = 1 \leq 3.$$

Moreover, the fuzzy transitivity condition implies that

$$T_R(x, z) = F(x, z) \geq \min\{F(x, y), F(y, z)\} = \min\{T_R(x, y), T_R(y, z)\}.$$

The acyclicity condition for F ensures that for any cycle, the minimal membership value is zero; hence, the corresponding T_R values will also satisfy the neutrosophic acyclicity condition. Therefore, (D, R) is a valid Neutrosophic Directed Set. $\square$

2.3 | Plithogenic Directed Set

The definition of the Plithogenic Directed Set is provided below.

Definition 21 (Plithogenic Directed Set). Let (D, v, pdf, pCF) be a plithogenic set, where:

- D is a nonempty set.
- v is an attribute defined on D with a range Pv .
- $pdf : D \times Pv \rightarrow [0, 1]^s$ is the Degree of Appurtenance Function.
- $pCF : Pv \times Pv \rightarrow [0, 1]^t$ is the Degree of Contradiction Function, satisfying for all $a, b \in Pv$:

$$pCF(a, a) = 0 \quad \text{and} \quad pCF(a, b) = pCF(b, a).$$

A *plithogenic directed set* is a pair (D, R) where $R : D \times D \rightarrow [0, 1]^s$ is a plithogenic relation satisfying:

(1) *Adjusted Transitivity*: For all $x, y, z \in D$,

$$R(x, z) \geq \min\{R(x, y), R(y, z)\} - pCF(v(x), v(z)).$$

(2) *Acyclicity*: For any finite cycle $x_1, x_2, \dots, x_n$ in D ,

$$\min\{R(x_1, x_2), R(x_2, x_3), \dots, R(x_n, x_1)\} \leq pCF(v(x_1), v(x_1)) = 0.$$

Example 22. Let $D = \{a, b, c\}$ with an attribute v taking values in $\{1, 2\}$. Suppose:

$$pdf(x, v(x)) = 0.9 \quad \text{for all } x \in D,$$

and let $pCF(1, 2) = 0.1$ with $pCF(1, 1) = pCF(2, 2) = 0$. Define the plithogenic relation R by:

$$R(a, b) = 0.8, \quad R(b, c) = 0.7, \quad R(a, c) = 0.7.$$

Then (D, R) forms a plithogenic directed set.

Theorem 23 (Plithogenic Reduction Under Low Contradiction). *In a plithogenic directed set (D, R) , if $pCF(v(x), v(z)) \approx 0$ for all $x, z \in D$, then the plithogenic directed set satisfies:*

$$R(x, z) \geq \min\{R(x, y), R(y, z)\},$$

i.e., it behaves like a fuzzy directed set.

Proof: If $pCF(v(x), v(z)) \approx 0$ for all $x, z \in D$, then the adjusted transitivity condition simplifies to:

$$R(x, z) \geq \min\{R(x, y), R(y, z)\},$$

which is the standard fuzzy transitivity condition. □

Theorem 24. *Let (D, R) be a Neutrosophic Directed Set with*

$$R(x, y) = (T_R(x, y), I_R(x, y), F_R(x, y)).$$

Assume that the plithogenic parameters (i.e., the attribute function v , the Degree of Appurtenance Function pdf , and the Degree of Contradiction Function pCF) on D are chosen such that $pCF(v(x), v(y)) \equiv 0$ for all $x, y \in D$. Then, the relation R satisfies the plithogenic transitivity condition:

$$R(x, z) \geq \min\{R(x, y), R(y, z)\} - pCF(v(x), v(z)),$$

which reduces to the neutrosophic transitivity condition. Thus, every Neutrosophic Directed Set is a special case of a Plithogenic Directed Set.

Proof: Given that $pCF(v(x), v(z)) = 0$ for all $x, z \in D$, the plithogenic transitivity condition simplifies to:

$$R(x, z) \geq \min\{R(x, y), R(y, z)\}.$$

Since the Neutrosophic Directed Set (D, R) already satisfies

$$T_R(x, z) \geq \min\{T_R(x, y), T_R(y, z)\},$$

the plithogenic directed relation R (with the additional parameters incorporated trivially via $pCF \equiv 0$) meets the required condition. Hence, the Neutrosophic Directed Set is subsumed by the definition of a Plithogenic Directed Set. $\square$

2.4 | Fuzzy MultiDirected Set

The definition of the Fuzzy MultiDirected Set is provided below.

Definition 25 (Fuzzy MultiDirected Set). A *fuzzy multi-directed set* is a quintuple

$$(M, R, i, t, \mu)$$

where:

- M is a nonempty set of nodes with a fuzzy membership function $\mu : M \rightarrow [0, 1]$.
- R is a set of directed relations (or edges) between nodes.
- $i : R \rightarrow M$ and $t : R \rightarrow M$ are the initial (source) and terminal (target) functions, respectively.
- For each relation $r \in R$, there is an associated fuzzy degree $\delta(r) \in [0, 1]$ satisfying

$$\delta(r) \leq \min\{\mu(i(r)), \mu(t(r))\}.$$

Example 26. Let $M = \{a, b, c\}$ with $\mu(a) = 0.9$, $\mu(b) = 0.8$, and $\mu(c) = 0.85$. Define relations $R = \{r_1, r_2\}$ by:

$$\begin{aligned} i(r_1) &= a, & t(r_1) &= b, & \delta(r_1) &= 0.7, \\ i(r_2) &= b, & t(r_2) &= c, & \delta(r_2) &= 0.6. \end{aligned}$$

Then (M, R, i, t, μ) is a fuzzy multi-directed set.

Theorem 27 (Fuzzy Chain Connectivity). In a fuzzy multi-directed set (M, R, i, t, μ) , if for every pair of nodes $u, v \in M$ there exists a chain $r_1, r_2, \dots, r_k$ such that

$$\min\{\delta(r_1), \delta(r_2), \dots, \delta(r_k)\} > 0,$$

then the set is fuzzy connected.

Proof: The existence of such a chain implies that there is a continuous path with nonzero fuzzy relation degree between any two nodes. Hence, the overall fuzzy connectivity is ensured. $\square$

Theorem 28. Let (M, R, i, t) be a MultiDirected Set, where R is a set of directed relations with initial and terminal functions i and t . Also, let (D, F) be a Fuzzy Directed Set. Define a Fuzzy MultiDirected Set as a quintuple

$$(M, R, i, t, \mu)$$

where $\mu : M \rightarrow [0, 1]$ is a fuzzy membership function on nodes and each relation $r \in R$ has an associated fuzzy degree $\delta(r) \in [0, 1]$ with

$$\delta(r) \leq \min\{\mu(i(r)), \mu(t(r))\}.$$

Then, the Fuzzy MultiDirected Set generalizes:

- (1) The MultiDirected Set (when μ is the crisp indicator function taking values in $\{0, 1\}$).
- (2) The Fuzzy Directed Set (by identifying the underlying set D with M and $F(x, y)$ with the fuzzy degree of the unique relation from x to y).

Proof: If we set $\mu(x) = 1$ for all $x \in M$ and assume that each directed relation is unique (or its multiplicity is one), then the condition $\delta(r) \leq \min\{\mu(i(r)), \mu(t(r))\} = 1$ is automatically satisfied, and the structure reduces to a classical MultiDirected Set. Alternatively, if the underlying structure is given by a Fuzzy Directed Set (D, F) , we can consider each edge in F as a relation r with $\delta(r) = F(x, y)$ and identify $M = D$ with the membership function $\mu(x) \equiv 1$ (or the same as the fuzzy set on nodes, if provided). Thus, the Fuzzy MultiDirected Set encompasses both the crisp multidirected structure and the fuzzy directed structure. $\square$

Theorem 29 (Sup–Min Transitivity). *Let (M, R, i, t, μ) be a fuzzy multi-directed set. For any two relations $r, s \in R$ with $t(r) = i(s)$, define their composition $r \circ s$ by*

$$i(r \circ s) = i(r), \quad t(r \circ s) = t(s), \quad \delta(r \circ s) = \min\{\delta(r), \delta(s)\}.$$

Then

$$\delta(r \circ s) \leq \min\{\mu(i(r \circ s)), \mu(t(r \circ s))\},$$

and for any three composable relations $r, s, u \in R$,

$$(r \circ s) \circ u = r \circ (s \circ u),$$

with

$$\delta((r \circ s) \circ u) = \min\{\delta(r), \delta(s), \delta(u)\} = \delta(r \circ (s \circ u)).$$

Hence composition is associative and closed in R .

Proof: Since $t(r) = i(s)$, we have

$$\delta(r \circ s) = \min\{\delta(r), \delta(s)\} \leq \min\{\mu(i(r)), \mu(t(r)), \mu(t(s))\} = \min\{\mu(i(r \circ s)), \mu(t(r \circ s))\},$$

where we used $\delta(r) \leq \mu(i(r)) \wedge \mu(t(r))$ and similarly for s . For associativity, observe

$$\delta((r \circ s) \circ u) = \min\{\delta(r \circ s), \delta(u)\} = \min\{\min\{\delta(r), \delta(s)\}, \delta(u)\} = \min\{\delta(r), \delta(s), \delta(u)\},$$

and likewise $\delta(r \circ (s \circ u))$ yields the same minimum. Equality of initial and terminal endpoints follows from the definitions, proving associativity and closure. $\square$

Theorem 30 (–Cut Yields a Crisp MultiDirected Subset). *Let (M, R, i, t, μ) be a fuzzy multi-directed set and fix $\alpha \in (0, 1]$. Define the α –cut node set*

$$M_\alpha = \{x \in M : \mu(x) \geq \alpha\},$$

and the α –cut relation set

$$R_\alpha = \{r \in R : \delta(r) \geq \alpha \text{ and } i(r), t(r) \in M_\alpha\}.$$

Then $(M_\alpha, R_\alpha, i|_{R_\alpha}, t|_{R_\alpha})$ is an ordinary (crisp) multi-directed set.

Proof: By construction, every relation $r \in R_\alpha$ satisfies

$$\delta(r) \geq \alpha \implies \mu(i(r)) \geq \alpha \text{ and } \mu(t(r)) \geq \alpha,$$

so $i(r), t(r) \in M_\alpha$. The initial and terminal functions clearly restrict to maps $R_\alpha \rightarrow M_\alpha$. Since all membership and relation-degrees are now at least $\alpha > 0$, the remaining multi-directed set axioms hold in the ordinary (crisp) sense: there are well-defined source and target for each edge, and no further fuzzy conditions are needed. Thus $(M_\alpha, R_\alpha, i, t)$ is a classical multi-directed set. $\square$

Theorem 31 (Bound on Fuzzy Path Strength). *In (M, R, i, t, μ) , for any path $p = r_1 r_2 \cdots r_k$ (where $t(r_j) = i(r_{j+1})$), define its strength*

$$\delta(p) = \min_{1 \leq j \leq k} \{\delta(r_j)\}.$$

Then for any two nodes $u, v \in M$, the maximum fuzzy connectivity

$$\Delta(u, v) = \max_{p: u \rightarrow v} \delta(p)$$

satisfies

$$\Delta(u, v) \leq \min\{\mu(u), \mu(v)\}.$$

Moreover, Δ defines a fuzzy reachability relation on M .

Proof: Each path p from u to v begins at u and ends at v , so for each edge r_j on p ,

$$\delta(r_j) \leq \min\{\mu(i(r_j)), \mu(t(r_j))\},$$

in particular $\delta(r_1) \leq \mu(u)$ and $\delta(r_k) \leq \mu(v)$. Hence

$$\delta(p) = \min_j \{\delta(r_j)\} \leq \min\{\mu(u), \mu(v)\}.$$

Taking the maximum over all such paths yields the stated bound. Symmetry is not assumed since the set is directed, but Δ inherits reflexivity ($\Delta(u, u) = \mu(u)$) and the usual sup-min composition law for transitive closure, making Δ a fuzzy preorder (reachability) on M . $\square$

2.5 | Neutrosophic MultiDirected Set

The definition of the Neutrosophic MultiDirected Set is provided below.

Definition 32 (Neutrosophic MultiDirected Set). A *neutrosophic multi-directed set* is a structure

$$(M, R, i, t, T, I, F)$$

where:

- M is a nonempty set of nodes.
- R is a set of relations.
- $i : R \rightarrow M$ and $t : R \rightarrow M$ assign the initial and terminal nodes, respectively.
- $T, I, F : M \cup R \rightarrow [0, 1]$ are functions representing the truth, indeterminacy, and falsity memberships, satisfying:

$$0 \leq T(x) + I(x) + F(x) \leq 3 \quad \text{for all } x \in M \cup R.$$

For each relation $r \in R$, we require

$$T(r) \leq \min\{T(i(r)), T(t(r))\}.$$

Example 33. Let $M = \{a, b\}$ with $T(a) = 0.9$, $I(a) = 0.05$, $F(a) = 0.05$ and $T(b) = 0.8$, $I(b) = 0.1$, $F(b) = 0.1$. Let $R = \{r\}$ with $i(r) = a$, $t(r) = b$ and

$$T(r) = 0.8, I(r) = 0.1, F(r) = 0.1.$$

Then (M, R, i, t, T, I, F) is a neutrosophic multi-directed set.

Theorem 34 (Neutrosophic Consistency). In a neutrosophic multi-directed set (M, R, i, t, T, I, F) , if for every relation $r \in R$

$$T(r) > 0 \quad \text{and} \quad I(r) + F(r) < 1,$$

then the structure behaves similarly to a crisp multi-directed set.

Proof: The conditions ensure that the truth membership is dominant and the uncertainty (indeterminacy and falsity) is sufficiently low. Consequently, the effective connectivity determined by R is primarily governed by T , thereby mimicking the behavior of a crisp multi-directed set. $\square$

Theorem 35. Let (M, R, i, t) be a MultiDirected Set and let (D, R') be a Neutrosophic Directed Set with

$$R'(x, y) = (T_R(x, y), I_R(x, y), F_R(x, y)).$$

A Neutrosophic MultiDirected Set is defined as a structure

$$(M, R, i, t, T, I, F)$$

where $T, I, F : M \cup R \rightarrow [0, 1]$ satisfy

$$0 \leq T(x) + I(x) + F(x) \leq 3 \quad \text{for all } x \in M \cup R,$$

and for each relation $r \in R$ with $i(r) = x$ and $t(r) = y$,

$$T(r) \leq \min\{T(x), T(y)\}.$$

By assigning trivial multiplicity (or identifying $M = D$) and by setting the node functions consistently, the Neutrosophic MultiDirected Set generalizes both the classical MultiDirected Set (when $T(x) = 1, I(x) = F(x) = 0$) and the Neutrosophic Directed Set.

Proof: If the neutrosophic membership functions are chosen such that for every node $x \in M$, $T(x) = 1$ and $I(x) = F(x) = 0$, then the structure (M, R, i, t, T, I, F) reduces to a standard MultiDirected Set. Alternatively, if we consider M to be the universe D of a Neutrosophic Directed Set and for each relation r between x and y we set $T(r) = T_R(x, y)$, $I(r) = I_R(x, y)$, and $F(r) = F_R(x, y)$, then the structure exactly recovers the Neutrosophic Directed Set. Thus, the Neutrosophic MultiDirected Set unifies both frameworks. $\square$

2.6 | Plithogenic MultiDirected Set

The definition of the Plithogenic MultiDirected Set is provided below.

Definition 36 (Plithogenic MultiDirected Set). A *plithogenic multi-directed set* is a structure

$$(M, R, i, t, v, pdf, pCF)$$

where:

- M is a nonempty set of nodes endowed with an attribute v taking values in a set Pv .
- $pdf : M \times Pv \rightarrow [0, 1]^s$ is the Degree of Appurtenance Function for nodes.
- R is a set of relations.
- $i : R \rightarrow M$ and $t : R \rightarrow M$ are the initial and terminal node functions.
- $pCF : Pv \times Pv \rightarrow [0, 1]^t$ is the Degree of Contradiction Function, satisfying $pCF(a, a) = 0$ and $pCF(a, b) = pCF(b, a)$ for all $a, b \in Pv$.

For each relation $r \in R$ connecting nodes x and y , the effective degree of the relation is governed by the node degrees $pdf(x, v(x))$ and $pdf(y, v(y))$, adjusted by $pCF(v(x), v(y))$.

Example 37. Let $M = \{a, b\}$ with an attribute v taking values in $\{1, 2\}$. Suppose:

$$pdf(a, 1) = 0.9, \quad pdf(b, 2) = 0.85,$$

and

$$pCF(1, 2) = 0.1 \quad \text{with} \quad pCF(1, 1) = pCF(2, 2) = 0.$$

Let $R = \{r\}$ with $i(r) = a$ and $t(r) = b$. Then the effective degree of the relation is approximately

$$\min\{0.9, 0.85\} - 0.1 = 0.75,$$

indicating a strong connection. Thus, $(M, R, i, t, v, pdf, pCF)$ is a plithogenic multi-directed set.

Theorem 38 (Plithogenic Reduction). In a plithogenic multi-directed set $(M, R, i, t, v, pdf, pCF)$, if $pCF(v(x), v(y)) \approx 0$ for all $x, y \in M$, then the structure reduces to a fuzzy multi-directed set.

Proof: When $pCF(v(x), v(y)) \approx 0$ for all $x, y \in M$, the effect of the contradiction is negligible. Consequently, the effective degree of the relation is determined solely by pdf , and the plithogenic multi-directed set behaves equivalently to a fuzzy multi-directed set. $\square$

Theorem 39. Let (M, R, i, t) be a MultiDirected Set and let (D, R_p) be a Plithogenic Directed Set defined on D with an attribute function v , a Degree of Appurtenance Function pdf , and a Degree of Contradiction Function pCF . A Plithogenic MultiDirected Set is defined as

$$(M, R, i, t, v, pdf, pCF),$$

where the plithogenic functions pdf and pCF are incorporated into the relation structure on M . By appropriately choosing v , pdf , and pCF (for example, by setting $pCF \equiv 0$ to recover the standard case), the Plithogenic MultiDirected Set reduces to:

- (1) A classical MultiDirected Set when the plithogenic parameters are trivial.
- (2) A Plithogenic Directed Set when the underlying multidirected structure contains a unique relation between any two nodes.

Hence, the Plithogenic MultiDirected Set generalizes both.

Proof: Assume that the plithogenic parameters are such that $pCF(v(x), v(y)) = 0$ for all $x, y \in M$. Then the adjusted transitivity condition in the Plithogenic Directed Set reduces to the standard condition, and the structure $(M, R, i, t, v, pdf, pCF)$ coincides with a classical MultiDirected Set if each pair (x, y) has a unique relation or with a Plithogenic Directed Set if the relation is determined by the plithogenic degrees. Thus, by the appropriate choice of the functions v , pdf , and pCF , the Plithogenic MultiDirected Set is seen as a unifying framework that contains both the multidirected and the plithogenic directed cases as special instances. $\square$

2.7 | Soft Directed Set

The definition of the Soft Directed Set is provided below.

Definition 40 (Soft Directed Set). Let D be a nonempty set (the universe of nodes) and A a set of parameters. A soft directed set over D is a pair $(\mathcal{F}, S)$ where $S \subseteq A$ and

$$\mathcal{F} : S \rightarrow \mathcal{P}(D \times D)$$

is a mapping such that for every parameter $\alpha \in S$:

- (1) *Transitivity:* If $(x, y) \in \mathcal{F}(\alpha)$ and $(y, z) \in \mathcal{F}(\alpha)$, then $(x, z) \in \mathcal{F}(\alpha)$ for all $x, y, z \in D$.
- (2) *Directedness:* For every $x, y \in D$, there exists some $z \in D$ such that $(x, z) \in \mathcal{F}(\alpha)$ and $(y, z) \in \mathcal{F}(\alpha)$.

Example 41. Let $D = \{1, 2, 3\}$ and $A = \{a, b\}$ with $S = A$. Define:

$$\mathcal{F}(a) = \{(1, 2), (2, 3), (1, 3)\}$$

and

$$\mathcal{F}(b) = \{(2, 3), (3, 1), (2, 1)\}.$$

For parameter a , note that transitivity holds since $(1, 2)$ and $(2, 3)$ imply $(1, 3) \in \mathcal{F}(a)$. Similarly, one can verify the directedness condition for each parameter.

Theorem 42 (Union Theorem for Soft Directed Sets). Let $(\mathcal{F}, S)$ be a soft directed set over D and assume that for every $x, y \in D$ there exists $\alpha \in S$ and $z \in D$ such that $(x, z) \in \mathcal{F}(\alpha)$ and $(y, z) \in \mathcal{F}(\alpha)$. Then the union

$$R = \bigcup_{\alpha \in S} \mathcal{F}(\alpha)$$

defines a directed relation on D .

Proof: Let $x, y \in D$. By the hypothesis, there exists $\alpha \in S$ and $z \in D$ such that $(x, z) \in \mathcal{F}(\alpha)$ and $(y, z) \in \mathcal{F}(\alpha)$. Since $\mathcal{F}(\alpha) \subseteq R$, it follows that (x, z) and (y, z) belong to R . Hence, R satisfies the directedness property. $\square$

Theorem 43 (Parameter Restriction). Let $(\mathcal{F}, S)$ be a soft directed set over D , and let $T \subseteq S$ be any nonempty subset of parameters. Define

$$\mathcal{F}|_T : T \longrightarrow \mathcal{P}(D \times D), \quad \mathcal{F}|_T(\alpha) = \mathcal{F}(\alpha) \quad (\alpha \in T).$$

Then $(\mathcal{F}|_T, T)$ is itself a soft directed set over D .

Proof: For each $\alpha \in T$, $\mathcal{F}|_T(\alpha) = \mathcal{F}(\alpha)$ remains transitive and directed by hypothesis. Thus the two axioms of Definition [Soft Directed Set] hold unchanged on T , so $(\mathcal{F}|_T, T)$ is a soft directed set. $\square$

Theorem 44 (Intersection Transitivity). *Let $(\mathcal{F}, S)$ be a soft directed set over D . Then the relation*

$$R_{\cap} = \bigcap_{\alpha \in S} \mathcal{F}(\alpha)$$

is transitive on D .

Proof: Suppose $(x, y) \in R_{\cap}$ and $(y, z) \in R_{\cap}$. By definition of intersection, for every $\alpha \in S$ both (x, y) and (y, z) lie in $\mathcal{F}(\alpha)$. Since each $\mathcal{F}(\alpha)$ is transitive, $(x, z) \in \mathcal{F}(\alpha)$ for every α . Hence $(x, z) \in R_{\cap}$, proving transitivity. $\square$

Theorem 45 (Transitive-Closure Soft Directed Set). *Let $(\mathcal{F}, S)$ be a soft directed set over D , and set*

$$R = \bigcup_{\alpha \in S} \mathcal{F}(\alpha),$$

and let R^+ be the transitive closure of R . Define a single-parameter mapping

$$\mathcal{G} : \{\gamma\} \longrightarrow \mathcal{P}(D \times D), \quad \mathcal{G}(\gamma) = R^+.$$

Then $(\mathcal{G}, \{\gamma\})$ is a soft directed set over D .

Proof: By construction R^+ is transitive. Moreover, since R is directed (Theorem [Union Theorem for Soft Directed Sets]), for any $x, y \in D$ there exists z with $(x, z), (y, z) \in R \subseteq R^+$. Hence R^+ satisfies directedness as well. Therefore $(\mathcal{G}, \{\gamma\})$ meets both axioms and is a soft directed set. $\square$

2.8 | Hypersoft Directed Set

The definition of the HyperSoft Directed Set is provided below.

Definition 46 (Hypersoft Directed Set). Let D be a nonempty set and let $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_m$ be attribute domains. Define the parameter space

$$\mathcal{C} = \mathcal{A}_1 \times \mathcal{A}_2 \times \dots \times \mathcal{A}_m.$$

A *hypersoft directed set* over D is a pair $(G, \mathcal{C})$ where

$$G : \mathcal{C} \rightarrow \mathcal{P}(D \times D)$$

assigns to each $\gamma \in \mathcal{C}$ a subset $G(\gamma) \subseteq D \times D$ that forms a directed relation on D (i.e., it is transitive and satisfies the directedness condition as in the soft directed set).

Example 47. Let $D = \{1, 2, 3, 4\}$ and consider two attribute domains $\mathcal{A}_1 = \{\text{red}, \text{blue}\}$ and $\mathcal{A}_2 = \{\text{high}, \text{low}\}$. Then the parameter space is

$$\mathcal{C} = \{(\text{red}, \text{high}), (\text{red}, \text{low}), (\text{blue}, \text{high}), (\text{blue}, \text{low})\}.$$

Define, for instance,

$$G(\text{red}, \text{high}) = \{(1, 2), (2, 3), (1, 3)\},$$

and assign appropriate directed relations for the other parameters. Each $G(\gamma)$ must satisfy transitivity and directedness.

Theorem 48 (Parameter-Subset Restriction). *Let $(G, \mathcal{C})$ be a hypersoft directed set over D , and let $\mathcal{C}' \subseteq \mathcal{C}$ be any nonempty subset of the parameter space. Define*

$$G|_{\mathcal{C}'} : \mathcal{C}' \longrightarrow \mathcal{P}(D \times D), \quad G|_{\mathcal{C}'}(\gamma) = G(\gamma) \quad (\gamma \in \mathcal{C}').$$

Then $(G|_{\mathcal{C}'}, \mathcal{C}')$ is a hypersoft directed set over D .

Proof: For each $\gamma \in \mathcal{C}'$, $G|_{\mathcal{C}'}(\gamma) = G(\gamma)$ remains transitive and directed by hypothesis. Hence the two defining properties of Definition [Hypersoft Directed Set] hold unchanged on $\mathcal{C}'$. $\square$

Theorem 49 (Union of Hypersoft Relations). *Let $(G, \mathcal{C})$ be a hypersoft directed set over D . Define the global relation*

$$R = \bigcup_{\gamma \in \mathcal{C}} G(\gamma).$$

Then R is a directed relation on D (i.e. transitive and satisfies directedness).

Proof: Directedness: For any $x, y \in D$, since each $G(\gamma)$ is directed, there exists some γ and $z \in D$ with $(x, z), (y, z) \in G(\gamma) \subseteq R$. *Transitivity:* If $(x, y), (y, z) \in R$, then there exist $\gamma_1, \gamma_2 \in \mathcal{C}$ with $(x, y) \in G(\gamma_1)$ and $(y, z) \in G(\gamma_2)$. By directedness of $G(\gamma_1)$ and $G(\gamma_2)$, there is some w so that $(x, w), (z, w)$ lie in each corresponding relation; taking a common parameter (e.g. choose $\gamma = \gamma_1$ and use transitivity in $G(\gamma)$ if $(y, z) \in G(\gamma)$, or refine by using the union structure) one shows $(x, z) \in R$. More directly, since every $G(\gamma)$ is transitive, any two edges in the same $G(\gamma)$ compose to give a third; and if they lie in different $G(\gamma)$, directedness ensures a “bridge” edge which by transitivity yields $(x, z) \in R$. Thus R is transitive. $\square$

Theorem 50 (Intersection Transitivity). *Let $(G, \mathcal{C})$ be a hypersoft directed set over D . Then the relation*

$$R_{\cap} = \bigcap_{\gamma \in \mathcal{C}} G(\gamma)$$

is transitive on D .

Proof: If $(x, y), (y, z) \in R_{\cap}$, then for every γ , both pairs lie in $G(\gamma)$. By transitivity of each $G(\gamma)$, we have $(x, z) \in G(\gamma)$ for all γ , hence $(x, z) \in R_{\cap}$. $\square$

Theorem 51 (Projection to a Soft Directed Set). *Fix an index $1 \leq k \leq m$, and let $\pi_k : \mathcal{C} \rightarrow \mathcal{A}_k$ be the projection onto the k th attribute. Define*

$$\mathcal{F} : \mathcal{A}_k \rightarrow \mathcal{P}(D \times D), \quad \mathcal{F}(a) = \bigcup_{\substack{\gamma \in \mathcal{C} \\ \pi_k(\gamma) = a}} G(\gamma), \quad a \in \mathcal{A}_k.$$

Then $(\mathcal{F}, \mathcal{A}_k)$ is a soft directed set over D .

Proof: Transitivity: If $(x, y), (y, z) \in \mathcal{F}(a)$, then there exist γ_1, γ_2 with $\pi_k(\gamma_i) = a$ and $(x, y) \in G(\gamma_1), (y, z) \in G(\gamma_2)$. Since $G(\gamma_1)$ and $G(\gamma_2)$ are each transitive, and both belong to the same union defining $\mathcal{F}(a)$, we conclude $(x, z) \in \mathcal{F}(a)$. Directedness: For any x, y , choose any γ and z such that $(x, z), (y, z) \in G(\gamma)$; then $\pi_k(\gamma) = a$ for some a , so $(x, z), (y, z) \in \mathcal{F}(a)$. Thus $\mathcal{F}(a)$ satisfies both soft-directed axioms for each a . $\square$

2.9 | Soft MultiDirected Set

The definition of the Soft MultiDirected Set is provided below.

Definition 52 (Soft MultiDirected Set). Let M be a nonempty set of nodes and A be a set of parameters. A *soft multidirected set* over M is a pair $(\mathcal{F}, S)$ where $S \subseteq A$ and

$$\mathcal{F} : S \rightarrow \{f : M \times M \rightarrow \mathbb{N}_0\}$$

assigns to each parameter $\alpha \in S$ a function $f_{\alpha} : M \times M \rightarrow \mathbb{N}_0$ that represents the multiplicity of directed relations between nodes. Moreover, for every $\alpha \in S$, the function f_{α} is assumed to satisfy a generalized directedness condition, for example:

$$\forall x, y \in M, \exists z \in M \text{ such that } f_{\alpha}(x, z) > 0 \text{ and } f_{\alpha}(y, z) > 0.$$

Example 53. Let $M = \{a, b, c\}$ and $A = \{p, q\}$ with $S = A$. Define $\mathcal{F}(p) : M \times M \rightarrow \mathbb{N}_0$ by:

$$\mathcal{F}(p)(a, b) = 1, \quad \mathcal{F}(p)(b, c) = 2, \quad \mathcal{F}(p)(a, c) = 1,$$

and zero elsewhere. For parameter q , define a different multiplicity function (for instance, swapping some relations). These functions represent the number of directed relations between nodes under different parameters.

Theorem 54 (Reduction to Classical MultiDirected Set). *If for a soft multidirected set $(\mathcal{F}, S)$ the membership functions are crisp in the sense that for each $\alpha \in S$, $f_\alpha(x, y) \in \{0, 1\}$ for all $x, y \in M$, then the soft multidirected set reduces to a classical multidirected set.*

Proof: When each $f_\alpha(x, y)$ is either 0 or 1, the function f_α defines a unique directed relation (with no multiple edges). Hence, for each parameter, the structure is equivalent to a directed set. The soft multidirected set, being a collection of such relations parameterized by S , is then equivalent to the union of classical multidirected sets. $\square$

Theorem 55 (Parameter Restriction). *Let $(\mathcal{F}, S)$ be a soft multidirected set over M , and let $T \subseteq S$ be nonempty. Define*

$$\mathcal{F}|_T : T \longrightarrow \{f : M \times M \rightarrow \mathbb{N}_0\}, \quad \mathcal{F}|_T(\alpha) = \mathcal{F}(\alpha) \quad (\alpha \in T).$$

Then $(\mathcal{F}|_T, T)$ is a soft multidirected set over M .

Proof: Since each f_α for $\alpha \in T$ already satisfies the multiplicity-directedness condition, restricting the parameter set to T preserves nonemptiness and the generalized directedness for every $\alpha \in T$. Hence $(\mathcal{F}|_T, T)$ meets the definition of a soft multidirected set. $\square$

Theorem 56 (Union Aggregation). *Let $(\mathcal{F}, S)$ be a soft multidirected set over M . Define the aggregated relation*

$$F(x, y) = \max_{\alpha \in S} \mathcal{F}(\alpha)(x, y), \quad x, y \in M.$$

If for every $x, y \in M$ there exists $\alpha \in S$ with $\mathcal{F}(\alpha)(x, y) > 0$, then (M, F) is a classical multidirected set.

Proof: By hypothesis $F(x, y) > 0$ for each pair (x, y) . Moreover, for any x, y there is α and z with $\mathcal{F}(\alpha)(x, z) > 0$ and $\mathcal{F}(\alpha)(y, z) > 0$. Since

$$F(x, z) \geq \mathcal{F}(\alpha)(x, z) > 0, \quad F(y, z) \geq \mathcal{F}(\alpha)(y, z) > 0,$$

the directedness condition holds for F . Thus F defines a valid multidirected relation on M . $\square$

Theorem 57 (Sum Aggregation). *Let $(\mathcal{F}, S)$ be a soft multidirected set over M . Define*

$$S(x, y) = \sum_{\alpha \in S} \mathcal{F}(\alpha)(x, y), \quad x, y \in M.$$

If for every $x, y \in M$ there exists $\alpha \in S$ with $\mathcal{F}(\alpha)(x, y) > 0$, then (M, S) is a classical multidirected set.

Proof: Because at least one $\mathcal{F}(\alpha)(x, y)$ is positive, $S(x, y) > 0$. For any x, y , pick α and z with $\mathcal{F}(\alpha)(x, z) > 0$, $\mathcal{F}(\alpha)(y, z) > 0$. Then

$$S(x, z) = \sum_{\beta} \mathcal{F}(\beta)(x, z) \geq \mathcal{F}(\alpha)(x, z) > 0, \quad S(y, z) \geq \mathcal{F}(\alpha)(y, z) > 0.$$

Hence S satisfies the multidirectedness property. $\square$

Theorem 58 (Intersection Aggregation). *Let $(\mathcal{F}, S)$ be a soft multidirected set over M . Define*

$$I(x, y) = \min_{\alpha \in S} \mathcal{F}(\alpha)(x, y), \quad x, y \in M.$$

If for every $x, y \in M$ there exists $\alpha \in S$ with $\mathcal{F}(\alpha)(x, y) > 0$, then (M, I) is a classical multidirected set.

Proof: Since each $\mathcal{F}(\alpha)(x, y) \geq 0$ and one is positive, $I(x, y) > 0$. For any x, y , choose α and z with $\mathcal{F}(\alpha)(x, z) > 0$, $\mathcal{F}(\alpha)(y, z) > 0$. Then

$$I(x, z) = \min_{\beta} \mathcal{F}(\beta)(x, z) \geq \mathcal{F}(\alpha)(x, z) > 0, \quad I(y, z) \geq \mathcal{F}(\alpha)(y, z) > 0,$$

so the directedness holds for I . $\square$

Theorem 59 (Transitive Closure). *Let $(\mathcal{F}, S)$ be a soft multidirected set over M . For each $\alpha \in S$, define the transitive-closure function*

$$\mathcal{F}^+(\alpha)(x, y) = \begin{cases} 1, & \text{if there is a chain } x = r_0 \rightarrow r_1 \rightarrow \dots \rightarrow r_k = y \text{ with } \mathcal{F}(\alpha)(r_{i-1}, r_i) > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then $(\mathcal{F}^+, S)$ is a soft multidirected set over M .

Proof: By construction each $\mathcal{F}^+(\alpha)$ is nonzero exactly on reachable pairs, ensuring > 0 for all (x, y) . Transitivity follows from path-concatenation, and directedness is inherited since each original $\mathcal{F}(\alpha)$ was directed. Thus $\mathcal{F}^+(\alpha)$ meets the generalized directedness condition for every α . $\square$

2.10 | Hypersoft MultiDirected Set

The definition of the HyperSoft MultiDirected Set is provided below.

Definition 60 (Hypersoft MultiDirected Set). *Let M be a nonempty set and let $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_m$ be attribute domains. Define the parameter space*

$$\mathcal{C} = \mathcal{A}_1 \times \mathcal{A}_2 \times \dots \times \mathcal{A}_m.$$

A hypersoft multidirected set over M is a pair $(G, \mathcal{C})$ where

$$G : \mathcal{C} \rightarrow \{f : M \times M \rightarrow \mathbb{N}_0\}$$

assigns to each $\gamma \in \mathcal{C}$ a function $f_\gamma : M \times M \rightarrow \mathbb{N}_0$ that represents the multiplicity of directed relations between nodes in M . For each γ , the function f_γ must satisfy a generalized directedness condition analogous to that of a multidirected set.

Example 61. Let $M = \{x, y, z\}$ and consider two attribute domains: $\mathcal{A}_1 = \{\text{low}, \text{high}\}$ and $\mathcal{A}_2 = \{\text{red}, \text{blue}\}$. Then, the parameter space is

$$\mathcal{C} = \{(\text{low}, \text{red}), (\text{low}, \text{blue}), (\text{high}, \text{red}), (\text{high}, \text{blue})\}.$$

Define, for instance,

$$G(\text{high}, \text{red})(x, y) = 2, \quad G(\text{high}, \text{red})(y, z) = 1, \quad G(\text{high}, \text{red})(x, z) = 1,$$

with all other values zero. Similar functions can be defined for other parameters in $\mathcal{C}$, each representing a multidirected relation.

Theorem 62 (Parameter Union in Hypersoft MultiDirected Sets). *Let $(G, \mathcal{C})$ be a hypersoft multidirected set over M . Define*

$$F(x, y) = \max_{\gamma \in \mathcal{C}} G(\gamma)(x, y)$$

for $x, y \in M$. If for every $x, y \in M$ there exists some $\gamma \in \mathcal{C}$ such that $G(\gamma)(x, y) > 0$, then $F : M \times M \rightarrow \mathbb{N}_0$ defines a multidirected relation on M .

Proof: For each $x, y \in M$, the condition ensures that there exists at least one $\gamma \in \mathcal{C}$ with $G(\gamma)(x, y) > 0$, so $F(x, y) > 0$. The function F is the pointwise maximum over all parameters and thus inherits the generalized directedness property from the functions $G(\gamma)$. Hence, F defines a multidirected relation on M . $\square$

Theorem 63 (Parameter Restriction). *Let $(G, \mathcal{C})$ be a hypersoft multidirected set over M , and let $\mathcal{C}' \subseteq \mathcal{C}$ be any nonempty subset. Define*

$$G|_{\mathcal{C}'} : \mathcal{C}' \longrightarrow \{f : M \times M \rightarrow \mathbb{N}_0\}, \quad G|_{\mathcal{C}'}(\gamma) = G(\gamma) \quad (\gamma \in \mathcal{C}').$$

Then $(G|_{\mathcal{C}'}, \mathcal{C}')$ is itself a hypersoft multidirected set over M .

Proof: For each $\gamma \in \mathcal{C}'$, the function $G|_{\mathcal{C}'}(\gamma) = G(\gamma)$ still assigns a multiplicity function $M \times M \rightarrow \mathbb{N}_0$ satisfying the multidirected-set axioms by hypothesis. Hence the restriction inherits both the nonemptiness of the parameter space and the generalized directedness condition for each parameter. $\square$

Theorem 64 (Union–Aggregation). *Let $(G, \mathcal{C})$ be a hypersoft multidirected set over M . Define the aggregated multiplicity*

$$F(x, y) = \max_{\gamma \in \mathcal{C}} G(\gamma)(x, y), \quad x, y \in M.$$

If for every $x, y \in M$ there is some $\gamma \in \mathcal{C}$ with $G(\gamma)(x, y) > 0$, then the single–parameter map

$$\widetilde{G} : \{*\} \longrightarrow \{f : M \times M \rightarrow \mathbb{N}_0\}, \quad \widetilde{G}(*) = F$$

defines a hypersoft multidirected set $(\widetilde{G}, \{*\})$. In particular, F is a valid multidirected relation on M .

Proof: By construction $F(x, y) > 0$ for each pair. Moreover, for any x, y there exists γ and z such that $G(\gamma)(x, z) > 0$ and $G(\gamma)(y, z) > 0$ (directedness at γ), hence

$$F(x, z) \geq G(\gamma)(x, z) > 0, \quad F(y, z) \geq G(\gamma)(y, z) > 0.$$

Thus F satisfies the multidirectedness condition. The parameter space $\{*\}$ is nonempty, so $(\widetilde{G}, \{*\})$ is hypersoft multidirected. $\square$

Theorem 65 (Sum–Aggregation). *Let $(G, \mathcal{C})$ be a hypersoft multidirected set over M . Define*

$$S(x, y) = \sum_{\gamma \in \mathcal{C}} G(\gamma)(x, y), \quad x, y \in M.$$

If for every $x, y \in M$ there exists $\gamma \in \mathcal{C}$ with $G(\gamma)(x, y) > 0$, then the map

$$\overline{G} : \{\diamond\} \longrightarrow \{f : M \times M \rightarrow \mathbb{N}_0\}, \quad \overline{G}(\diamond) = S$$

defines a hypersoft multidirected set $(\overline{G}, \{\diamond\})$. In particular, S is a valid multidirected relation on M .

Proof: Since at least one term in the sum is positive, $S(x, y) > 0$. Directedness follows because for any x, y there is some γ and z with $G(\gamma)(x, z), G(\gamma)(y, z) > 0$; then

$$S(x, z) = \sum_{\beta} G(\beta)(x, z) \geq G(\gamma)(x, z) > 0, \quad S(y, z) \geq G(\gamma)(y, z) > 0.$$

Thus S meets the multidirectedness requirement and $(\overline{G}, \{\diamond\})$ is hypersoft multidirected. $\square$

Theorem 66 (Transitive Closure). *Let $(G, \mathcal{C})$ be a hypersoft multidirected set over M . For each $\gamma \in \mathcal{C}$, define the transitive-closure multiplicity*

$$G^+(\gamma)(x, y) = \begin{cases} 1, & \text{if there exists a chain } r_1, \dots, r_k \text{ in } G(\gamma) \text{ from } x \text{ to } y, \\ 0, & \text{otherwise.} \end{cases}$$

Then $(G^+, \mathcal{C})$ is also a hypersoft multidirected set over M .

Proof: By definition $G^+(\gamma)$ has an edge (x, y) whenever there is a directed path in $G(\gamma)$, so it contains the original relations and is thus directed. Transitivity of $G^+(\gamma)$ is immediate from path-concatenation, and nonemptiness of $\mathcal{C}$ remains unchanged. Hence each $G^+(\gamma)$ satisfies the multidirected-set axioms, proving that $(G^+, \mathcal{C})$ is hypersoft multidirected. $\square$

3 | Conclusion

In this paper, we introduced and examined the newly defined concepts of Fuzzy MultiDirected Sets, Neutrosophic MultiDirected Sets, and Plithogenic MultiDirected Sets. As a direction for future research, potential extensions to frameworks such as OffSet (cf. [29, 30]) are also suggested.

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Data Availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare that there is no conflict of interest in the research.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

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