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Fourth-Derivative Hybrid Block Methods for Initial Value Problems in ODEs: A Comprehensive Study

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Abstract

In this paper, we introduce a new and improved way to solve a common type of mathematical problem known as an initial value problem. We developed two new methods, called Fourth-Derivative Hybrid Block Methods (FDHBMs), which are both highly accurate and reliable. After testing our methods, we found they are very stable, meaning they can be trusted for complex, long-term calculations. When we compared them to existing techniques, our new methods performed significantly better. One of them, which we call FDHBM1, was particularly impressive; it's so precise that its accuracy is near-perfect, and it's much better at minimizing errors than existing methods. Ultimately, our work shows that this new approach is a powerful and dependable tool for solving these kinds of differential equations. The convergence and stability properties were also examined. It was discovered that the new method compared favorably with other existing methods in the selected literature.

Keywords: Fourth Derivative; Hybrid Block Method; Initial Value Problems; Differential Equations.

1 | Introduction

Solving a certain class of mathematical problems known as Initial Value Problems (IVPs) in Ordinary Differential Equations (ODEs) is crucial across many fields, from engineering and physics to biology. Since finding their exact solutions is often impossible, especially for complex systems, researchers rely on numerical methods to get reliable approximations [1] and [2].

To overcome these challenges, hybrid block methods have emerged as promising alternatives. Hybrid block methods are characterized by their ability to compute multiple solution points within a single integration step, thereby improving efficiency and reducing computational costs [3] and [4]. They also allow for higher-order accuracy through the inclusion of additional derivative information [5]. In particular, the incorporation of higher derivatives, such as the fourth derivative, enhances precision and stability by providing more information about the local behavior of the solution [6].

Several studies have contributed to the development of hybrid block methods. [7] constructed Adams-type hybrid block methods associated with Chebyshev polynomials to solve first-order ODEs, while [8] developed a fifth-stage, fourth-order Runge–Kutta scheme for IVPs. [4] developed an Implicit One-Step Block Hybrid Third-Derivative Method for the Direct Solution of Initial Value Problems of Second-Order Ordinary Differential Equations, [9] constructed a One-Step Embedded Hybrid Block Method for Solving First Order Stiff Initial Value Problems of Ordinary Differential Equations, [10] proposed continuous fourth-derivative



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block methods for stiff systems, demonstrating their effectiveness in achieving high accuracy. Furthermore, on hybrid block methods, [11] developed a One-step hybrid block method for directly solving fifth-order initial value problems of ordinary differential equations. More recently, [12] advanced a one-step Obrechhoff-type hybrid block method that confirmed the potential of hybrid block frameworks in solving IVPs efficiently.

A review of existing scholarly research shows that one-step hybrid block methods have demonstrated superior accuracy compared to many established techniques for solving initial value problems in ordinary differential equations. Nevertheless, deriving these methods often involves considerable complexity, prompting the need for approaches that are both simpler and more computationally efficient. In response to this gap, the present paper introduces a novel and efficient numerical algorithm specifically developed for solving initial value problems. The specific objectives of this study are:

- a. Derive One and Two Step Fourth-Derivative Hybrid Block Methods for the solution of Initial Value Problems in ODEs.
- b. Present the analysis of the basic properties of the methods, such as Order, Consistency, and Zero Stability.
- c. Present numerical examples and compare with existing methods, give a profile graph of numerical examples as well as their error profile graph, a discussion on findings, and where FDHBMs can be applied for future research.

This paper, therefore, contributes to the growing body of work on hybrid block methods by advancing fourth-derivative-based schemes that provide reliable and highly accurate solutions for IVPs in ODEs. This research work contains different sections. Section 1 is basically the introduction aspect of the work, Section 2 shows the development of the methods, Section 3 presents the analysis of the basic properties of the methods, Section 4 contains the numerical examples, discussion findings, and where FDHBMs can be applied for future researchers, and finally, Section 5 gives a conclusion of the research work.

2 | Derivation of One- and Two-Step Hybrid Block Methods

Consider the initial value problem defined on $t \in [0, T]$ by

$$y'(t) = f(t, y(t)), y(t_0) = y_0$$

$f(t, y)$ a general continuous nonlinear function.

We proposed a k-step continuous fourth derivative hybrid block method by obtaining the main method for one step, by considering the form.

$$y_{n+k} = \sum_{j=0}^{k-1} \alpha_j y_{n+j} + \sum_{i=1}^l h^i \sum_{\eta=0}^k (\beta_{\eta i} f_{n+i} + \beta_{\eta i} g_{n+i} + \beta_{\eta i} p_{n+i} + \beta_{\eta i} q_{n+i}), \eta = 1, 2, 3, 4, k = 1, \\ l = 4. \quad (2.1)$$

For the solution of first-order initial value problems on the interval from x_n to x_{n+1} where α_j and $\beta_{\eta i}$ are the coefficients, k is the step number, and h is the step length. Interpolation and collation methods are used in the derivation of this method. We assume that $y_{n+i} = y(x_n + ih)$ is the numerical approximation to the analytical solution $y(x_{n+i})$, $y'_{n+i} = f(x_{n+i})$ is an approximation to $y'(x_{n+i})$, $g_{n+i} = f'(x_{n+i}, y(x_{n+i}))$, $p_{n+i} = f''(x_{n+i}, y(x_{n+i}))$, and $q_{n+i} = f'''(x_{n+i}, y(x_{n+i}))$. We need a continuous representation of a one-step method to approximate the exact solution $y(x)$. We expand Eq. (2.1) to be;

$$\begin{aligned}
y_{n+1} = & \alpha_0 y_0 + h \left(\beta_{10} f_n + \beta_{1\frac{1}{2}} f_{n+\frac{1}{2}} + \beta_{11} f_{n+1} \right) + h^2 \left(\beta_{20} g_n + \beta_{2\frac{1}{2}} g_{n+\frac{1}{2}} + \beta_{21} g_{n+1} \right) \\
& + h^3 \left(\beta_{30} p_n + \beta_{3\frac{1}{2}} p_{n+\frac{1}{2}} + \beta_{31} p_{n+1} \right) \\
& + h^4 \left(\beta_{40} q_n + \beta_{4\frac{1}{2}} q_{n+\frac{1}{2}} + \beta_{41} q_{n+1} \right) \quad (2.2)
\end{aligned}$$

Using Taylor series expansion to expand individual terms in (2.2) and upon substitution of the expansions back into (2.2), the matrix form $Mx = N$ is obtained from the expression:

$$y_{n+b} = y(x_n + bh) = y(x_n) + bhy'(x_n) + \frac{(bh)^2}{2!} y''(x_n) + \frac{(bh)^3}{3!} y'''(x_n) + \dots \quad (2.3).$$

And the coefficients of $y^{(r)}(x_n)$ are equated and solved simultaneously to obtain the main method scheme as:

$$\begin{aligned}
y_{n+1} = & \alpha_0 y_0 + h \left(\frac{174876341}{1928474730} f_n + \frac{93323264}{623918295} f_{n+\frac{1}{4}} + \frac{839808}{1616615} f_{n+\frac{1}{2}} + \frac{93323264}{623918295} f_{n+\frac{3}{4}} + \right. \\
& \left. \frac{174876341}{1928474730} f_{n+1} \right) + h^2 \left(\frac{2752961}{831891060} g_n - \frac{561152}{24724035} g_{n+\frac{1}{4}} + 0 g_{n+\frac{1}{2}} + \frac{561152}{24724035} g_{n+\frac{3}{4}} - \right. \\
& \left. \frac{2752961}{831891060} g_{n+1} \right) + h^3 \left(\frac{2993}{51802740} p_n - \frac{275456}{1178512335} p_{n+\frac{1}{4}} + \frac{28528}{14549535} p_{n+\frac{1}{2}} - \frac{275456}{1178512335} p_{n+\frac{3}{4}} + \right. \\
& \left. \frac{2993}{51802740} p_{n+1} \right) + h^4 \left(\frac{7703}{18856197360} q_n - \frac{9472}{235702467} q_{n+\frac{1}{4}} + 0 q_{n+\frac{1}{2}} + \frac{9472}{235702467} q_{n+\frac{3}{4}} - \right. \\
& \left. \frac{7703}{18856197360} q_{n+1} \right) \quad (2.4)
\end{aligned}$$

With an additional scheme as:

$$\begin{aligned}
y_{n+\frac{1}{4}} = & \alpha_0 y_0 + h \left(\frac{25640687281697}{288878902640640} f_n - \frac{119665500707}{1277784668160} f_{n+\frac{1}{4}} + \frac{209952}{1616615} f_{n+\frac{1}{2}} + \right. \\
& \left. \frac{157532484643}{1277784668160} f_{n+\frac{3}{4}} + \frac{3507359413529}{2022152318484480} f_{n+1} \right) + h^2 \left(\frac{2761011139871}{872301000130560} g_n - \frac{8602204927}{202539294720} g_{n+\frac{1}{4}} - \right. \\
& \left. \frac{59049}{1146880} g_{n+\frac{1}{2}} - \frac{548801587}{40507858944} g_{n+\frac{3}{4}} - \frac{113779292897}{872301000130560} g_{n+1} \right) + h^3 \left(\frac{20497759259}{380233769287680} p_n - \right. \\
& \left. \frac{18102863843}{19308746096640} p_{n+\frac{1}{4}} + \frac{2795761}{3724680960} p_{n+\frac{1}{2}} + \frac{10383462883}{19308746096640} p_{n+\frac{3}{4}} + \frac{1335925733}{380233769287680} p_{n+1} \right) + \\
& h^4 \left(\frac{7339298063}{19772156002959360} q_n - \frac{8441859421}{154469968773120} q_{n+\frac{1}{4}} - \frac{729}{7340032} q_{n+\frac{1}{2}} - \frac{372341113}{30893993754624} q_{n+\frac{3}{4}} - \right. \\
& \left. \frac{19208155}{564918742941696} q_{n+1} \right) \quad (2.5)
\end{aligned}$$

$$\begin{aligned}
y_{n+\frac{1}{2}} = & \alpha_0 y_0 + h \left(\frac{131641071869}{1481068592640} f_n + \frac{37825696}{1871754885} f_{n+\frac{1}{4}} + \frac{419904}{1616615} f_{n+\frac{1}{2}} + \frac{242144096}{1871754885} f_{n+\frac{3}{4}} + \right. \\
& \left. \frac{2663958019}{1481068592640} f_{n+1} \right) + h^2 \left(\frac{2027891237}{638892334080} g_n - \frac{547160}{14834421} g_{n+\frac{1}{4}} - \frac{17}{280} g_{n+\frac{1}{2}} - \frac{1052344}{74172105} g_{n+\frac{3}{4}} - \right. \\
& \left. \frac{86382811}{638892334080} g_{n+1} \right) + h^3 \left(\frac{15076571}{278491530240} p_n - \frac{2815012}{3535537005} p_{n+\frac{1}{4}} + \frac{14264}{14549535} p_{n+\frac{1}{2}} + \right.
\end{aligned}$$

$$\begin{aligned} & \left(\frac{284092}{505076715} p_{n+\frac{3}{4}} + \frac{1013797}{278491530240} p_{n+1} \right) + h^4 \left(\frac{5405969}{14481559572480} q_n - \frac{74605}{1414214802} q_{n+\frac{1}{4}} + \frac{13}{120960} q_{n+\frac{1}{2}} - \right. \\ & \left. \frac{2539}{202030686} q_{n+\frac{3}{4}} - \frac{101987}{2896311914496} q_{n+1} \right) \end{aligned} \quad (2.6)$$

$$\begin{aligned} y_{n+\frac{3}{4}} = & \alpha_0 y_0 + h \left(\frac{27414079367}{308207943680} f_n + \frac{5120189}{194754560} f_{n+\frac{1}{4}} + \frac{629856}{1616615} f_{n+\frac{1}{2}} + \frac{6767077}{27822080} f_{n+\frac{3}{4}} + \right. \\ & \left. \frac{592337017}{308207943680} f_{n+1} \right) + h^2 \left(\frac{181129371}{56979619840} g_n - \frac{3356637}{92610560} g_{n+\frac{1}{4}} - \frac{59049}{1146880} g_{n+\frac{1}{2}} - \frac{261627}{13230080} g_{n+\frac{3}{4}} - \right. \\ & \left. \frac{11493159}{79771467776} g_{n+1} \right) + h^3 \left(\frac{28302867}{521582673920} p_n - \frac{20434203}{26486620160} p_{n+\frac{1}{4}} + \frac{500823}{413853440} p_{n+\frac{1}{2}} + \right. \\ & \left. \frac{18641691}{26486620160} p_{n+\frac{3}{4}} + \frac{2017773}{521582673920} p_{n+1} \right) + h^4 \left(\frac{10157607}{27122299043840} q_n - \frac{2213793}{42378592256} q_{n+\frac{1}{4}} - \right. \\ & \left. \frac{729}{7340032} q_{n+\frac{1}{2}} - \frac{3064869}{211892961280} q_{n+\frac{3}{4}} - \frac{202437}{5424459808768} q_{n+1} \right) \end{aligned} \quad (2.7)$$

Hence, Eqs. (2.4) to (2.7) is the hybrid block scheme for a one-step fourth-derivative hybrid block method.

To further determine the efficiency of the fourth derivative, we further developed a family of two-step fourth derivative hybrid block methods. This was achieved by applying the same procedure for developing a step fourth derivative step as presented, to have the main method as:

$$\begin{aligned} y_{n+2} = & \alpha_0 y_0 + h \left(\frac{174876341}{964237365} f_n + \frac{186646528}{623918295} f_{n+\frac{1}{2}} + \frac{1679616}{1616615} f_{n+1} + \frac{186646528}{623918295} f_{n+\frac{3}{2}} \right. \\ & \left. + \frac{174876341}{964237365} f_{n+2} \right) \\ & + h^2 \left(\frac{2752961}{207972765} g_n - \frac{2244608}{24724035} g_{n+\frac{1}{2}} - 0 g_{n+1} - \frac{2244608}{24724035} g_{n+\frac{3}{2}} \right. \\ & \left. - \frac{2752961}{207972765} g_{n+2} \right) \\ & + h^3 \left(\frac{5986}{12950685} p_n - \frac{2203648}{1178512335} p_{n+\frac{1}{2}} + \frac{228224}{14549535} p_{n+1} \right. \\ & \left. + \frac{2203648}{1178512335} p_{n+\frac{3}{2}} + \frac{5986}{12950685} p_{n+2} \right) \\ & + h^4 \left(\frac{7703}{1178512335} q_n - \frac{151552}{235702467} q_{n+\frac{1}{2}} + 0 q_{n+1} + \frac{151552}{235702467} q_{n+\frac{3}{2}} \right. \\ & \left. - \frac{7703}{1178512335} q_{n+2} \right) \end{aligned} \quad (2.8)$$

With additional method as:

$$\begin{aligned}
y_{n+\frac{1}{2}} = & \alpha_0 y_0 + h \left(\frac{25640687281697}{144439451320320} f_n - \frac{119665500707}{638892334080} f_{n+\frac{1}{2}} + \frac{419904}{1616615} f_{n+1} \right. \\
& + \frac{157532484643}{638892334080} f_{n+\frac{3}{2}} + \left. \frac{3507359413529}{1011076159242240} f_{n+2} \right) \\
& + h^2 \left(\frac{2761011139871}{218075250032640} g_n - \frac{8602204927}{50634823680} g_{n+\frac{1}{2}} - \frac{59049}{286720} g_{n+1} \right. \\
& - \frac{548801587}{10126964736} g_{n+\frac{3}{2}} - \left. \frac{113779292897}{218075250032640} g_{n+2} \right) \\
& + h^3 \left(\frac{20497759259}{47529221160960} p_n - \frac{18102863843}{2413593262080} p_{n+\frac{1}{2}} + \frac{2795761}{465585120} p_{n+1} \right. \\
& + \frac{10383462883}{2413593262080} p_{n+\frac{3}{2}} + \left. \frac{1335925733}{47529221160960} p_{n+2} \right) \\
& + h^4 \left(\frac{7339298063}{1235759750184960} q_n - \frac{8441859421}{9654373048320} q_{n+\frac{1}{2}} - \frac{729}{458752} q_{n+1} \right. \\
& - \frac{372341113}{1930874609664} q_{n+\frac{3}{2}} \\
& \left. - \frac{19208155}{35307421433856} q_{n+2} \right) \quad (2.9)
\end{aligned}$$

$$\begin{aligned}
y_{n+1} = & \alpha_0 y_0 + h \left(\frac{131641071869}{740534296320} f_n + \frac{75651392}{1871754885} f_{n+\frac{1}{2}} + \frac{839808}{1616615} f_{n+1} \right. \\
& + \frac{484288192}{1871754885} f_{n+\frac{3}{2}} + \left. \frac{2663958019}{740534296320} f_{n+2} \right) \\
& + h^2 \left(\frac{2027891237}{159723083520} g_n - \frac{2188640}{14834421} g_{n+\frac{1}{2}} - \frac{17}{70} g_{n+1} - \frac{4209376}{74172105} g_{n+\frac{3}{2}} \right. \\
& - \left. \frac{86382811}{159723083520} g_{n+2} \right) \\
& + h^3 \left(\frac{15076571}{34811441280} p_n - \frac{22520096}{3535537005} p_{n+\frac{1}{2}} + \frac{114112}{14549535} p_{n+1} \right. \\
& + \frac{2272736}{505076715} p_{n+\frac{3}{2}} + \left. \frac{1013797}{34811441280} p_{n+2} \right) \\
& + h^4 \left(\frac{5405969}{905097473280} q_n - \frac{596840}{707107401} q_{n+\frac{1}{2}} - \frac{13}{7560} q_{n+1} \right. \\
& - \frac{20312}{101015343} q_{n+\frac{3}{2}} \\
& \left. - \frac{101987}{181019494656} q_{n+2} \right) \quad (2.10)
\end{aligned}$$

$$\begin{aligned}
y_{n+\frac{3}{2}} = & \alpha_0 y_0 + h \left(\frac{2741407936}{154103971840} f_n + \frac{5120189}{97377280} f_{n+\frac{1}{2}} + \frac{1259712}{1616615} f_{n+1} + \frac{6767077}{13911040} f_{n+\frac{3}{2}} \right. \\
& \left. + \frac{592337017}{154103971840} f_{n+2} \right) \\
& + h^2 \left(\frac{181129371}{14244904960} g_n - \frac{3356637}{23152640} g_{n+\frac{1}{2}} - \frac{59049}{286720} g_{n+1} - \frac{261627}{3307520} g_{n+\frac{3}{2}} \right. \\
& \left. - \frac{11493159}{19942866944} g_{n+2} \right) \\
& + h^3 \left(\frac{28302867}{65197834240} p_n - \frac{20434203}{3310827520} p_{n+\frac{1}{2}} + \frac{500823}{51731680} p_{n+1} \right. \\
& \left. + \frac{18641691}{3310827520} p_{n+\frac{3}{2}} + \frac{2017773}{65197834240} p_{n+2} \right) \\
& + h^4 \left(\frac{10157607}{1695143690240} q_n - \frac{2213793}{2648662016} q_{n+\frac{1}{2}} - \frac{729}{458752} q_{n+1} \right. \\
& \left. - \frac{3064869}{13243310080} q_{n+\frac{3}{2}} \right. \\
& \left. - \frac{202437}{339028738048} q_{n+2} \right) \quad (2.11)
\end{aligned}$$

Eqs. (2.8) to (2.11) are the required two-step fourth-derivative hybrid block method.

3 | Properties of the Methods

3.1 | Order and Consistency

According to [4], we can define the operation R by the equation:

$$R[y(x): h] = \sum_{i=0}^k [\alpha_i y(x_n + ih) - h \beta_i y'(x_n + ih)] \quad (3.1)$$

$y(x)$ is an arbitrary test function that is continuously differentiable over the interval $[a, b]$. Expanding $y(x_n + ih)$ and $y'(x_n + ih)$ in a Taylor series about the point x_n , and then grouping terms according to powers of h and y , yields:

$$R[y(x): h] = C_0 y(x) + C_1 h y'(x) + C_2 h^2 y''(x) + C_3 h^3 y'''(x) + C_4 h^4 y^{(4)}(x) + \dots + C_r h^r y^{(r)}(x) \quad (3.2)$$

Where the constant coefficients $C_r, r = 0, 1, 2, 3, \dots$

Definition 3.1

According to [10], the methods in Eq. (2.1) have the order r if

$$R[y(x): h] = O(h^{r-1}), C_0 = C_1 = \dots = C_r = 0, C_{r+1} \neq 0. \quad (3.3)$$

Therefore, C_{r+1} is the error constant and $C_{r+1} h^{r+1} y^{(r+1)}(x_n)$ The principal local truncation error at the point x_n . It was established from our calculations that the one-step has order Nineteen with an error constant as

$$\left(-\frac{127}{10604876112341630185793126400000}, -\frac{7105519}{1216252031572236882748242080563200000}, \right. \\
\left. -\frac{127}{21209752224683260371586252800000}, -\frac{379}{61792004855572670972323430400000} \right)^T$$

While the two-step hybrid block scheme has order Nineteen as well, with error constants as:

$$\left(\begin{array}{c} -\frac{127}{5056798988505187123200000}, -\frac{7105519}{579954162393682900785561600000} \\ -\frac{127}{10113597977010374246400000}, -\frac{379}{29464723995004973875200000} \end{array} \right)^T$$

According to [12], a linear multi-step method is said to be consistent if it has order $r \geq 1$. This implies the methods are consistent.

3.2 | Zero Stability of the Methods

Definition 3.2 [12] noted that a hybrid block method is said to be zero-stable if the roots Z of the characteristic polynomial $\bar{r}(Z)$ Defined by:

$$\bar{r}(Z) = \det[ZA^0 - A'] \quad (3.4)$$

They further claim that Eq. (3.4) satisfies $|Z| \leq 1$ and every root with $|Z| = 1$ has multiplicity not exceeding two in the limit as $h \rightarrow 0$. To further examine the zero stability of the method, we put Eqs. (2.4) to (2.7) in matrix form as a hybrid block method to obtain:

$$\begin{aligned} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n+\frac{1}{4}} \\ y_{n+\frac{1}{2}} \\ y_{n+\frac{3}{4}} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n-\frac{3}{4}} \\ y_{n-\frac{1}{2}} \\ y_{n-\frac{1}{4}} \\ y_n \end{bmatrix} + \\ & h \left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{174876341}{1928474730} \\ \frac{25640687281697}{288878902640640} \\ \frac{131641071869}{1481068592640} \\ \frac{27414079367}{308207943680} \end{bmatrix} \begin{bmatrix} f_{n-\frac{3}{4}} \\ f_{n-\frac{1}{2}} \\ f_{n-\frac{1}{4}} \\ f_n \end{bmatrix} + \right. \\ & \left. - \begin{bmatrix} \frac{93323264}{623918295} & \frac{839808}{1616615} & \frac{93323264}{623918295} & \frac{174876341}{1928474730} \\ \frac{119665500707}{1277784668160} & \frac{209952}{1616615} & \frac{157532484643}{1277784668160} & \frac{3507359413529}{2022152318484480} \\ \frac{37825696}{1871754885} & \frac{419904}{1616615} & \frac{242144096}{1871754885} & \frac{2663958019}{1481068592640} \\ \frac{5120189}{194754560} & \frac{629856}{1616615} & \frac{6767077}{27822080} & \frac{592337017}{308207943680} \end{bmatrix} \begin{bmatrix} f_{n+\frac{1}{4}} \\ f_{n+\frac{1}{2}} \\ f_{n+\frac{3}{4}} \\ f_{n+1} \end{bmatrix} \right) \\ & + h^2 \left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{2752961}{831891060} \\ \frac{2761011139871}{872301000130560} \\ \frac{2027891237}{638892334080} \\ \frac{181129371}{56979619840} \end{bmatrix} \begin{bmatrix} g_{n-\frac{3}{4}} \\ g_{n-\frac{1}{2}} \\ g_{n-\frac{1}{4}} \\ g_n \end{bmatrix} \right) \\ & + h^2 \left(\begin{bmatrix} -\frac{561152}{24724035} & 0 & \frac{561152}{24724035} & -\frac{2752961}{831891060} \\ -\frac{8602204927}{202539294720} & -\frac{59049}{1146880} & -\frac{548801587}{40507858944} & -\frac{113779292897}{872301000130560} \\ -\frac{547160}{14834421} & -\frac{17}{280} & -\frac{1052344}{74172105} & -\frac{86382811}{638892334080} \\ -\frac{3356637}{92610560} & -\frac{59049}{1146880} & -\frac{261627}{13230080} & -\frac{11493159}{79771467776} \end{bmatrix} \begin{bmatrix} g_{n+\frac{1}{4}} \\ g_{n+\frac{1}{2}} \\ g_{n+\frac{3}{4}} \\ g_{n+1} \end{bmatrix} \right) \end{aligned}$$

$$\begin{aligned}
& +h^3 \begin{pmatrix} 0 & 0 & 0 & \frac{2993}{51802740} \\ 0 & 0 & 0 & \frac{20497759259}{380233769287680} \\ 0 & 0 & 0 & \frac{15076571}{278491530240} \\ 0 & 0 & 0 & \frac{28302867}{521582673920} \end{pmatrix} \begin{bmatrix} p_{n-\frac{3}{4}} \\ p_{n-\frac{1}{2}} \\ p_{n-\frac{1}{4}} \\ p_n \end{bmatrix} \\
& +h^3 \begin{bmatrix} -\frac{275456}{1178512335} & \frac{28528}{14549535} & -\frac{275456}{1178512335} & \frac{2993}{51802740} \\ -\frac{18102863843}{19308746096640} & \frac{2795761}{3724680960} & \frac{10383462883}{19308746096640} & \frac{1335925733}{380233769287680} \\ -\frac{2815012}{3535537005} & \frac{14264}{14549535} & \frac{284092}{505076715} & \frac{1013797}{278491530240} \\ -\frac{20434203}{26486620160} & \frac{500823}{413853440} & \frac{18641691}{26486620160} & \frac{2017773}{521582673920} \end{bmatrix} \begin{bmatrix} p_{n+\frac{1}{4}} \\ p_{n+\frac{1}{2}} \\ p_{n+\frac{3}{4}} \\ p_{n+1} \end{bmatrix} \\
& +h^4 \begin{pmatrix} 0 & 0 & 0 & \frac{7703}{18856197360} \\ 0 & 0 & 0 & \frac{7339298063}{19772156002959360} \\ 0 & 0 & 0 & \frac{5405969}{14481559572480} \\ 0 & 0 & 0 & \frac{10157607}{27122299043840} \end{pmatrix} \begin{bmatrix} q_{n-\frac{3}{4}} \\ q_{n-\frac{1}{2}} \\ q_{n-\frac{1}{4}} \\ q_n \end{bmatrix} + \\
& +h^4 \begin{bmatrix} -\frac{9472}{235702467} & 0 & \frac{9472}{235702467} & -\frac{7703}{18856197360} \\ -\frac{8441859421}{154469968773120} & -\frac{729}{7340032} & -\frac{372341113}{30893993754624} & -\frac{19208155}{564918742941696} \\ -\frac{74605}{1414214802} & \frac{13}{120960} & -\frac{2539}{202030686} & -\frac{101987}{2896311914496} \\ -\frac{2213793}{42378592256} & -\frac{729}{7340032} & -\frac{3064869}{211892961280} & -\frac{202437}{5424459808768} \end{bmatrix} \begin{bmatrix} q_{n+\frac{1}{4}} \\ q_{n+\frac{1}{2}} \\ q_{n+\frac{3}{4}} \\ q_{n+1} \end{bmatrix}
\end{aligned}$$

Following Eq. (3.4), we can deduce the first characteristic polynomial of the hybrid block method as:

$$\det(ZA^0 - A') = \left| Z \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right|$$

Hence, the roots of the polynomial are $z = 0, 0, 0, 1$. Thus, the hybrid block method is zero-stable.

4 | Numerical Experiments and Results

The following first-order initial value problems were numerically solved using Maple 18 software, and a comparison of accuracy in terms of error with previously existing methods. For better understanding, we assume the following:

- 1) **FDHBM1**: Fourth-Derivative Hybrid Block Method for Initial Value Problems in ODEs Scheme One.

- 2) **FDHBM2:** Fourth-Derivative Hybrid Block Method for Initial Value Problems in ODEs Scheme Two.
- 3) **Error** = Computed solution Minus Exact Solution.

Problem 1 (See Tables 1a and 1b, Figures 1 and 4)

$$y' = 0.5(1 - y), y(0) = 0.5, h = 0.1 \text{ with exact solution } y(x) = 1 - 0.5e^{-0.5x}$$

Table 1a. Comparison of computed results for problem 1.

x	Exact solution	Computed solution in FDHBM1	Computed solution in FDHBM2	Computed solution in [13]
0.1	0.52438528774964299546	0.52438528774964456866	0.52438528774900080572	0.52438528774960472804
0.2	0.54758129098202021342	0.54758129098202320638	0.54758129098079847388	0.54758129098194536511
0.3	0.56964601178747109638	0.56964601178747536687	0.56964601178572786449	0.56964601178736527269
0.4	0.59063462346100907066	0.59063462346101448695	0.59063462345879811937	0.59063462346087361956
0.5	0.61059960846429756588	0.61059960846430400604	0.61059960846166866347	0.61059960846413739010
0.6	0.62959088965914106696	0.62959088965914841825	0.62959088965614023978	0.62959088965895722513
0.7	0.64765595514064328282	0.64765595514065144104	0.64765595513731306186	0.64765595514044005788
0.8	0.66483997698218034963	0.66483997698218921858	0.66483997697856000201	0.66483997698195855368
0.9	0.68118592418911335343	0.68118592418912284439	0.68118592418523909960	0.68118592418887672320
1.0	0.69673467014368328820	0.69673467014369331940	0.69673467013958850572	0.69673467014343242661

Table 1b. Comparison of Errors for Problem 1.

x	Error in FDHBM1	Error in FDHBM2	Error in [13]
0.1	1.3555510 ⁻¹⁵	6.716890910 ⁻¹³	3.82674010 ⁻¹⁴
0.2	2.8430910 ⁻¹⁵	1.2347177710 ⁻¹²	7.48483010 ⁻¹⁴
0.3	4.1823210 ⁻¹⁵	1.7412303210 ⁻¹²	1.058240 10 ⁻¹³
0.4	5.3841510 ⁻¹⁵	2.1954001910 ⁻¹²	1.35451010 ⁻¹³
0.5	6.4588210 ⁻¹⁵	2.6011281810 ⁻¹²	1.60176010 ⁻¹³
0.6	7.4158410 ⁻¹⁵	2.9620590810 ⁻¹²	1.83842010 ⁻¹³
0.7	8.2641710 ⁻¹⁵	3.2815973910 ⁻¹²	2.03225010 ⁻¹³
0.8	9.0121010 ⁻¹⁵	3.5629221110 ⁻¹²	2.21796010 ⁻¹³
0.9	9.6674210 ⁻¹⁵	3.8090005810 ⁻¹²	2.36630010 ⁻¹³
1.0	1.02373910 ⁻¹⁴	4.0226016310 ⁻¹²	2.50862010 ⁻¹³

Problem 2 (See Tables 2a and 2b, Figures 2 and 5)

$$y' = y, y(0) = 1, h = 0.1 \text{ with exact solution } y(x) = e^x$$

Table 2a. Comparison of computed results for problem 2.

x	Exact solution	Computed solution in FDHBM1	Computed solution in FDHBM2	Computed solution in [8]
0.1	1.1051709180756476248	1.1051709180757845863	1.1051709180170341227	1.105170882161
0.2	1.2214027581601698339	1.2214027581604725656	1.2214027580306139580	1.221402678778
0.3	1.3498588075760031040	1.3498588075765049594	1.3498588073612310245	1.349858675979
0.4	1.4918246976412703178	1.4918246976420098325	1.4918246973247905095	1.491824503725
0.5	1.6487212707001281468	1.6487212707011497595	1.6487212702629227965	1.648721002812
0.6	1.8221188003905089749	1.8221188003918638428	1.8221187998106850088	1.822118445116
0.7	2.0137527074704765216	2.0137527074722234424	2.0137527067228711726	2.013752249391
0.8	2.2255409284924676046	2.2255409284946740572	2.2255409275482028161	2.225540349914
0.9	2.4596031111569496638	2.4596031111596929845	2.4596031099829289328	2.459602391801
1.0	2.7182818284590452354	2.7182818284624139445	2.7182818270173857142	2.718280945113

Table 2b. Comparison of Errors for Problem 2.

x	Error in FDHBM1	Error in FDHBM2	Error in [8]
0.1	1.03524010^{-13}	6.5394706110^{-11}	3.591418940010^{-8}
0.2	2.54840510^{-13}	1.40371501310^{-10}	7.938263380010^{-8}
0.3	4.36839810^{-13}	2.30395727410^{-10}	1.315970652410^{-7}
0.4	6.54302810^{-13}	3.37803176410^{-10}	1.939163294410^{-7}
0.5	9.12675210^{-13}	4.65254516110^{-10}	2.678883555010^{-7}
0.6	1.218157210^{-12}	6.15777790010^{-10}	3.552748977210^{-7}
0.7	1.5777996110^{-12}	7.92816261910^{-10}	4.580793917510^{-7}
0.8	1.999616110^{-12}	1.000282261010^{-9}	5.785783012710^{-7}
0.9	2.492706210^{-12}	1.242617760410^{-9}	7.193563895710^{-7}
1.0	3.067396210^{-12}	1.524862509510^{-9}	8.833463875910^{-7}

Problem 3 (See Tables 3a and 3b, Figures 3 and 6)

$$y' = -y, y(0) = 1, h = 0.1 \text{ with exact solution } y(x) = e^{-x}$$

Table 3a. Comparison of computed results for problem 3.

x	Exact solution	Computed solution in FDHBM1	Computed solution in FDHBM2	Computed solution in [7]
0.1	0.90483741803595957316	0.90483741803586884228	0.90483741807247663316	0.904837417881202
0.2	0.81873075307798185867	0.81873075307781766527	0.81873075314406586323	0.818730752939751
0.3	0.74081822068171786607	0.74081822068149501357	0.74081822077141078616	0.740818220548903
0.4	0.67032004603563930074	0.67032004603537044037	0.67032004614384931439	0.670320045918305
0.5	0.60653065971263342360	0.60653065971232932995	0.60653065983502401030	0.606530659599218
0.6	0.54881163609402643263	0.54881163609369624625	0.54881163622691873158	0.548811635994641
0.7	0.49658530379140951470	0.49658530379106095555	0.49658530393169642681	0.496585303694640
0.8	0.44932896411722159143	0.44932896411686114644	0.44932896426229227410	0.449328964033219
0.9	0.40656965974059911188	0.40656965974023219976	0.40656965988827266657	0.406569659658082
1.0	0.36787944117144232160	0.36787944117107343735	0.36787944131990960819	0.367879441100594

Table 3b. Comparison of Errors for Problem 3.

x	Error FDHBM1	Error FDHBM2	Error in [7]
0.1	8.38226710^{-14}	$3.739526844 \cdot 10^{-11}$	$1.5476 \cdot 10^{-10}$
0.2	1.621517810^{-13}	6.52671629510^{-11}	1.382310^{-10}
0.3	2.248138010^{-13}	8.74956861410^{-11}	1.328210^{-10}
0.4	2.740812110^{-13}	1.049025079910^{-10}	1.173310^{-10}
0.5	3.119359110^{-13}	1.182040199410^{-10}	1.134210^{-10}
0.6	3.401038510^{-13}	1.280239310610^{-10}	$9.9385 \cdot 10^{-11}$
0.7	3.600858510^{-13}	1.349044201710^{-10}	9.677010^{-11}
0.8	3.731848310^{-13}	1.393160049810^{-10}	8.400310^{-11}
0.9	3.805298310^{-13}	1.416662709510^{-10}	8.251710^{-11}
1.0	3.830973710^{-13}	1.423075880010^{-10}	7.084810^{-11}

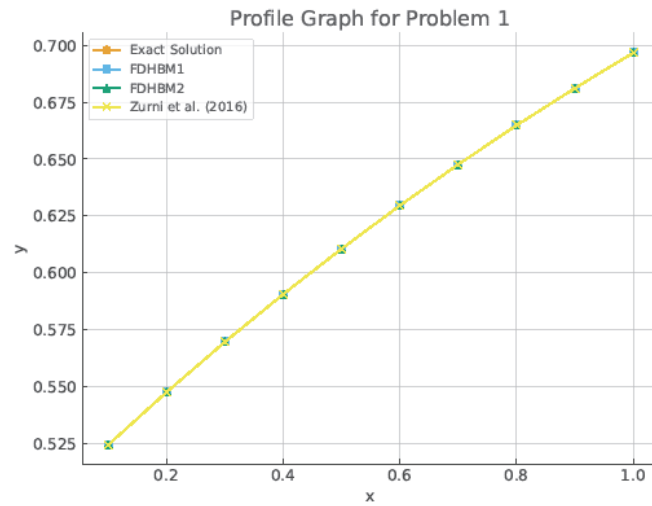


Figure 1. Profile graph for problem 1.

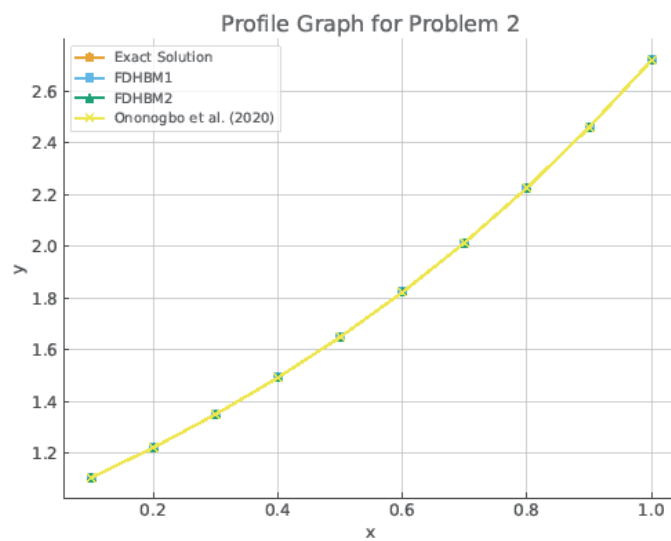


Figure 2. Profile graph for problem 2.

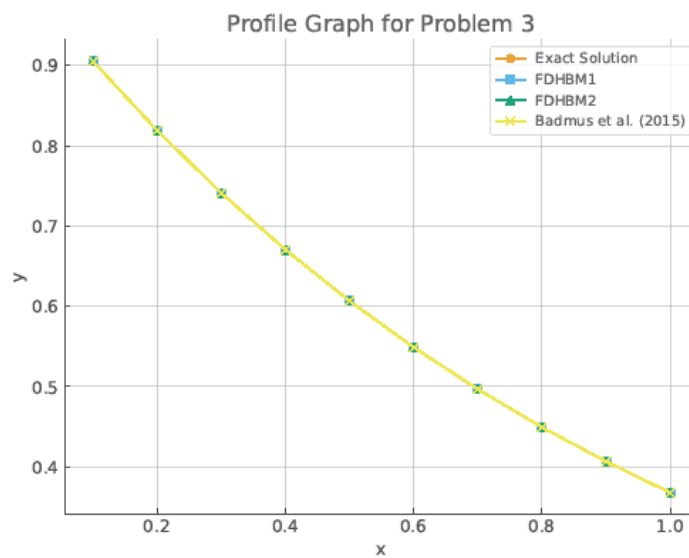


Figure 3. Profile graph for problem 3.

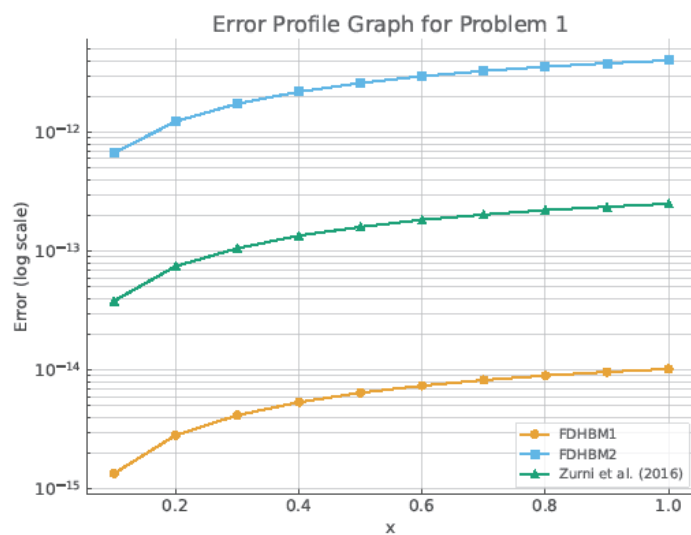


Figure 4. Error Profile graph for problem 1.

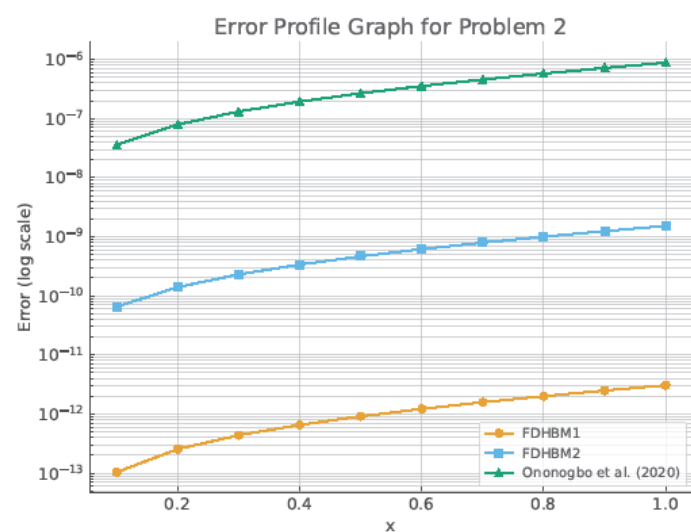


Figure 5. Error Profile graph for problem 2.

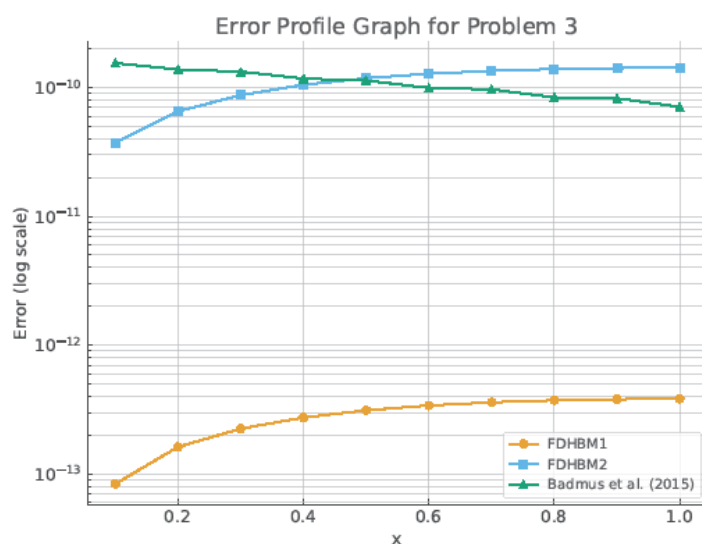


Figure 6. Error Profile graph for problem 3.

4.1 | Discussion of Results

Across all three problems, the FDHBM1 method consistently achieves the smallest errors and the closest agreement with the exact solutions. FDHBM2 performs reasonably well but with noticeably larger errors, while the previously reported methods [7, 8, 13] demonstrate comparatively lower accuracy. These results collectively confirm the superior reliability, stability, and precision of FDHBM1 for solving the class of problems considered in this study.

4.2 | Findings

The major findings are:

1) FDHBM1 Offers near Machine-Precision Accuracy

Across all test problems, FDHBM1 consistently produces the smallest errors, often in the order of 10^{-15} , the Limit of double-precision floating point computations. This makes it extremely suitable for scientific and engineering tasks requiring high accuracy.

2) FDHBM2 Performs Well but Shows Slightly Higher Errors

FDHBM2 accuracy ranges between 10^{-13} and 10^{-10} , still highly accurate but slightly less precise than FDHBM1. It remains superior to previously published classical methods.

3) Existing Methods Are Less Accurate

Methods from [7-8, 13] show significantly larger errors, for example, errors as high as 10^{-7} in the case of [8]. This demonstrates that the newly developed FDHBMs represent a major improvement over older approaches.

4) Both FDHBM1 and FDHBM2 Are Consistent and Zero-Stable

The theoretical analysis confirms that the proposed hybrid block methods meet the requirements for stable long-term numerical integration, making them dependable for real-world computations.

4.3 | Real World Applications of the Scheme

The highly accurate and stable nature of FDHBM1 and FDHBM2 enables their application across numerous scientific and engineering fields. Below are some practical examples where the findings can be applied for future research:

1) Engineering: Heat Transfer and Thermal Systems

Differential equations governing heat conduction, such as Newton's law of cooling or exponential decay models which require accurate numerical solutions.

2) Electrical and Electronics Engineering

Many circuits, including RC and RL circuits, are governed by first-order differential equations. FDHBM1 can be applied for charging or discharging of capacitors, Signal attenuation studies, and Transient analysis in digital and analog circuits. Accurate modeling improves design predictions, reduces component stress, and enhances power efficiency.

3) Population Growth and Epidemiology

Logistic and exponential models used in Population projection, Spread of infectious diseases, and Pharmaceutical drug absorption rates are all based on ODEs. FDHBM1 can model these systems more accurately for forecasting and intervention planning.

4) Control Systems and Robotics

First-order ODEs arise in Proportional Integral controllers, Actuator dynamics, and Autonomous system movement. Accurate numerical integration ensures stability and responsiveness in real-time control systems. In general, FDHBM1 is a major advancement in solving IVPs with extremely high precision and strong stability characteristics.

This makes the method suitable for any real-world problem governed by first-order ODEs, particularly where High accuracy is essential, long-term integration is needed, and where Classical methods fail, due to instability or error accumulation. Thus, the FDHBMs, especially FDHBM1, represent a practical and highly efficient tool for scientists, engineers, and mathematicians working with dynamic systems.

5 | Conclusion

In this study, Fourth-Derivative Hybrid Block Methods (FDHBMs) were proposed, analyzed, and applied to the numerical solution of first-order Initial Value Problems in Ordinary Differential Equations. The one-step and two-step FDHBM schemes derived through interpolation and collocation attained order nineteen with favorable error constants. The methods were shown to be consistent and zero-stable, satisfying the fundamental requirements of numerical integration techniques.

Numerical experiments further validated the efficiency of the proposed methods. FDHBM1, in particular, consistently produced results with errors approaching machine precision, outperforming FDHBM2 and existing methods from previous studies, including those of [7-8, 13]. The demonstrated superiority of FDHBM1 highlights the advantage of incorporating higher derivative information in hybrid block methods for improved accuracy and stability.

In Summary, the findings underscore the robustness of fourth-derivative hybrid block approaches as reliable tools for solving IVPs in ODEs. Future research may focus on extending these methods to stiff systems, partial differential equations, and real-world applications in science and engineering, as listed before, thereby broadening their applicability and impact.

Author Contribution

All authors contributed equally to this work.

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Data Availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data, but are available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare that there is no conflict of interest in the research.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

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