



Note of IndertermRough set and IndetermHyperRough set

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Abstract: Various frameworks have been proposed to address uncertainty, including Fuzzy Sets, Intuitionistic Fuzzy Sets, Picture Fuzzy Sets, Vague Sets, and Neutrosophic Sets. A Rough Set approximates uncertain or vague information by means of lower and upper bounds defined via equivalence classes within a universe. Rough Sets have been extensively studied in applications such as decision-making and machine learning. Another prominent approach to modeling uncertainty is the Soft Set, which has been extended in various directions—including the IndetermSoft Set and the IndetermHyperSoft Set—to capture indeterminate information. In this paper, we define and investigate the mathematical structures of IndetermRough Sets and IndetermHyperRough Sets, which generalize the classical Rough Set framework by incorporating indeterminacy into both relations and membership.

Keywords: Hyperstructure, Rough set, Hyperrough set, IndertermRough set

1 Preliminaries and Definitions

In this section, we present the key concepts and definitions essential for understanding the content of this paper. For a comprehensive background in set theory and related topics, readers may consult [1]. Unless otherwise stated, all sets considered in this paper are assumed to be finite and simple (i.e., without repeated elements).

1.1 Rough Set

A Rough Set approximates a subset using lower and upper bounds based on equivalence classes, capturing certainty and uncertainty in membership [2]. The definitions are provided below.

Definition 1 (Rough Set Approximation). [3] Let X be a non-empty universe of discourse, and let $R \subseteq X \times X$ be an equivalence relation (or indiscernibility relation) on X . The equivalence relation R partitions X into disjoint equivalence classes, denoted by $[x]_R$ for $x \in X$, where:

$$[x]_R = \{y \in X \mid (x, y) \in R\}.$$

For any subset $U \subseteq X$, the *lower approximation* $\underline{U}$ and the *upper approximation* $\overline{U}$ of U are defined as follows:

(1) *Lower Approximation* $\underline{U}$:

$$\underline{U} = \{x \in X \mid [x]_R \subseteq U\}.$$

The lower approximation $\underline{U}$ includes all elements of X whose equivalence classes are entirely contained within U . These are the elements that *definitely* belong to U .

(2) *Upper Approximation* $\overline{U}$:

$$\overline{U} = \{x \in X \mid [x]_R \cap U \neq \emptyset\}.$$

The upper approximation $\overline{U}$ contains all elements of X whose equivalence classes have a non-empty intersection with U . These are the elements that *possibly* belong to U .

The pair $(\underline{U}, \overline{U})$ forms the *rough set* representation of U , satisfying the relationship:

$$\underline{U} \subseteq U \subseteq \overline{U}.$$

1.2 HyperRough Set

The *HyperRough Set* is a concept that adapts the framework of the HyperSoft Set[4, 5] to Rough Set theory. Its formal definition is provided below.

Definition 2 (HyperRough Set). [6, 5] Let X be a non-empty finite universe, and let $T_1, T_2, \dots, T_n$ be n distinct attributes with respective domains $J_1, J_2, \dots, J_n$. Define the Cartesian product of these domains as:

$$J = J_1 \times J_2 \times \dots \times J_n.$$

Let $R \subseteq X \times X$ be an equivalence relation on X , where $[x]_R$ denotes the equivalence class of x under R .

A *HyperRough Set* over X is a pair (F, J) , where:

- $F : J \rightarrow \mathcal{P}(X)$ is a mapping that assigns a subset $F(a) \subseteq X$ to each attribute value combination $a = (a_1, a_2, \dots, a_n) \in J$.
- For each $a \in J$, the rough set $(\underline{F(a)}, \overline{F(a)})$ is defined as:

$$\underline{F(a)} = \{x \in X \mid [x]_R \subseteq F(a)\}, \quad \overline{F(a)} = \{x \in X \mid [x]_R \cap F(a) \neq \emptyset\}.$$

The *lower approximation* $\underline{F(a)}$ represents the set of elements in X whose equivalence classes are entirely contained within $F(a)$, while the *upper approximation* $\overline{F(a)}$ includes elements whose equivalence classes have a non-empty intersection with $F(a)$.

Additionally, the following properties hold:

- $\underline{F(a)} \subseteq \overline{F(a)}$ for all $a \in J$.
- If $F(a) = \emptyset$, then $\underline{F(a)} = \overline{F(a)} = \emptyset$.
- If $F(a) = X$, then $\underline{F(a)} = \overline{F(a)} = X$.

Example 3 (HyperRough Set in E-Commerce Customer Segmentation). Let

$$X = \{c_1, c_2, c_3, c_4\}$$

be a set of four customers. Define an equivalence relation R on X by “similar purchase history”:

$$R = \{(c_1, c_1), (c_1, c_2), (c_2, c_1), (c_2, c_2), \\ (c_3, c_3), (c_3, c_4), (c_4, c_3), (c_4, c_4)\}.$$

Hence the equivalence classes are $[c_1]_R = \{c_1, c_2\}$ and $[c_3]_R = \{c_3, c_4\}$.

We consider two customer attributes:

$$J_1 = \{\text{Silver}, \text{Gold}\}, \quad J_2 = \{\text{North}, \text{South}\},$$

so that

$$J = J_1 \times J_2 = \{(\text{Silver}, \text{North}), (\text{Silver}, \text{South}), (\text{Gold}, \text{North}), (\text{Gold}, \text{South})\}.$$

Define the mapping

$$F : J \rightarrow \mathcal{P}(X)$$

by

$$\begin{aligned} F(\text{Silver}, \text{North}) &= \{c_1, c_3\}, \\ F(\text{Silver}, \text{South}) &= \{c_2\}, \\ F(\text{Gold}, \text{North}) &= \{c_1, c_4\}, \\ F(\text{Gold}, \text{South}) &= \emptyset. \end{aligned}$$

For each attribute combination $(t, r) \in J$, the HyperRough approximations are

$$\underline{F}(t, r) = \{x \in X \mid [x]_R \subseteq F(t, r)\}, \quad \overline{F}(t, r) = \{x \in X \mid [x]_R \cap F(t, r) \neq \emptyset\}.$$

A direct computation yields:

$$\begin{aligned} \underline{F}(\text{Silver}, \text{North}) &= \emptyset, & \overline{F}(\text{Silver}, \text{North}) &= \{c_1, c_2, c_3, c_4\}, \\ \underline{F}(\text{Silver}, \text{South}) &= \emptyset, & \overline{F}(\text{Silver}, \text{South}) &= \{c_1, c_2\}, \\ \underline{F}(\text{Gold}, \text{North}) &= \emptyset, & \overline{F}(\text{Gold}, \text{North}) &= \{c_1, c_2, c_3, c_4\}, \\ \underline{F}(\text{Gold}, \text{South}) &= \emptyset, & \overline{F}(\text{Gold}, \text{South}) &= \emptyset. \end{aligned}$$

Thus the HyperRough Set (F, J) captures, for each customer segment (tier $\times$ region), the set of customers who definitely responded to a marketing campaign (lower approximation) and those who possibly responded (upper approximation).

1.3 Soft Set and Hypersoft set

A Soft Set provides a simplified approach to parameterized decision modeling by mapping attributes, or parameters, to subsets of a universal set, thereby addressing uncertainty in a straightforward manner [7, 8, 9]. Expanding on this idea, a Hypersoft Set facilitates enhanced multi-attribute decision analysis by mapping combinations of multiple attributes to subsets of a universal set [4, 10, 11, 12].

Definition 4 (Soft Set). [7, 8] Let U be a universal set and A be a set of attributes. A soft set over U is a pair $(\mathcal{F}, S)$, where $S \subseteq A$ and $\mathcal{F} : S \rightarrow \mathcal{P}(U)$. Here, $\mathcal{P}(U)$ denotes the power set of U . Mathematically, a soft set is represented as:

$$(\mathcal{F}, S) = \{(\alpha, \mathcal{F}(\alpha)) \mid \alpha \in S, \mathcal{F}(\alpha) \in \mathcal{P}(U)\}.$$

Each $\alpha \in S$ is called a parameter, and $\mathcal{F}(\alpha)$ is the set of elements in U associated with α .

Definition 5 (Hypersoft Set). [4] Let U be a universal set, and let $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_m$ be attribute domains. Define $\mathcal{C} = \mathcal{A}_1 \times \mathcal{A}_2 \times \dots \times \mathcal{A}_m$, the Cartesian product of these domains. A hypersoft set over U is a pair $(G, \mathcal{C})$, where $G : \mathcal{C} \rightarrow \mathcal{P}(U)$. The hypersoft set is expressed as:

$$(G, \mathcal{C}) = \{(\gamma, G(\gamma)) \mid \gamma \in \mathcal{C}, G(\gamma) \in \mathcal{P}(U)\}.$$

For an m -tuple $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_m) \in \mathcal{C}$, where $\gamma_i \in \mathcal{A}_i$ for $i = 1, 2, \dots, m$, $G(\gamma)$ represents the subset of U corresponding to the combination of attribute values $\gamma_1, \gamma_2, \dots, \gamma_m$.

1.4 IndetermSoft set and IndetermHyperSoft Set

An *IndetermSoft Set* models uncertainty by associating attribute values with subsets of a universal set where indeterminacy exists in attributes or images. And an *IndetermHyperSoft Set* generalizes IndetermSoft Sets to multiple distinct attributes, mapping combinations of indeterminate attribute values to subsets of a universal set.

$$F : A_1 \times A_2 \times \dots \times A_n \rightarrow P(H),$$

where indeterminacy can occur in any A_i , $P(H)$, or $F(a_1, a_2, \dots, a_n)$.

Definition 6 (IndetermSoft set). [13, 14] Let U be a universe of discourse, $H \subseteq U$ a non-empty subset, and $P(H)$ the powerset of H . Let A be the set of attribute values for an attribute a . A function $F : A \rightarrow P(H)$ is called an *IndetermSoft Set* if at least one of the following conditions holds:

- (1) A has some indeterminacy.
- (2) $P(H)$ has some indeterminacy.
- (3) There exists at least one $v \in A$ such that $F(v)$ is indeterminate (unclear, uncertain, or not unique).
- (4) Any two or all three of the above conditions.

An IndermSoft Set is represented mathematically as:

$$F : A \rightarrow H(\cap, \cup, \oplus, \neg),$$

where $H(\cap, \cup, \oplus, \neg)$ represents a structure closed under the IndermSoft operators.

Definition 7 (IndermHyperSoft Set). [13, 14] Let U be a universe of discourse, $H \subseteq U$ a non-empty subset, and $P(H)$ the powerset of H . Let $a_1, a_2, \dots, a_n$ ($n \geq 1$) be n distinct attributes, with attribute values $A_1, A_2, \dots, A_n$, such that $A_i \cap A_j = \emptyset$ for $i \neq j$. The pair $(F, A_1 \times A_2 \times \dots \times A_n)$, where

$$F : A_1 \times A_2 \times \dots \times A_n \rightarrow P(H),$$

is called an *IndermHyperSoft Set* if:

- (1) Any A_i or $P(H)$ exhibits indeterminacy.
- (2) For $(a_1, a_2, \dots, a_n) \in A_1 \times A_2 \times \dots \times A_n$, $F(a_1, a_2, \dots, a_n)$ is indeterminate.

Example 8 (IndermHyperSoft Set for Job Candidate Shortlisting). Let

$$U = \{\text{Alice, Bob, Carol, Dave}\}$$

be the universe of all job applicants. Set $H = U$, so $\mathcal{P}(H)$ is the powerset of these four candidates.

We consider two distinct attributes:

$$A_1 = \{\text{BSc, MSc}\}, \quad A_2 = \{\text{Software, DataScience}\},$$

where BSc/MSc is the highest degree held and Software/DataScience is the primary skill area.

Define the (possibly indeterminate) mapping

$$F : A_1 \times A_2 \rightarrow \mathcal{P}(H)$$

by listing “def” for definite membership and “?” for indeterminate membership:

$$\begin{aligned} F(\text{BSc, Software}) &= \{\text{Alice (def), Bob (?), Carol (?)}\}, \\ F(\text{BSc, DataScience}) &= \{\text{Carol (def), Dave (?)}\}, \\ F(\text{MSc, Software}) &= \{\text{Bob (def), Dave (?)}\}, \\ F(\text{MSc, DataScience}) &= \{\text{Alice (def), Carol (?)}\}. \end{aligned}$$

Here, for example, “Bob (?)” in $F(\text{BSc, Software})$ means it is not yet confirmed whether Bob graduated with a BSc in Software Engineering. Similarly, “Dave (?)” in $F(\text{BSc, DataScience})$ indicates Dave’s undergraduate major is still being verified.

Because some values of $F(a_1, a_2)$ are indeterminate, the powerset $\mathcal{P}(H)$ exhibits indeterminacy. Thus $(F, A_1 \times A_2)$ satisfies the conditions of an IndermHyperSoft Set.

Interpretation. This structure models the situation where applicants are grouped by (degree, skill) combinations, but some applicants’ profiles are not fully verified. Downstream decision processes can then compute intersections, unions, or other soft-set operations while respecting these indeterminate memberships.

2 Results of This Paper

The results of this paper are presented as follows.

2.1 IndetermRough Set and IndetermHyperRough Set

An IndetermRough Set extends Rough Set theory by introducing indeterminate equivalence relations and uncertain element memberships. It defines lower and upper approximations that account for partial knowledge, allowing for more flexible handling of uncertainty. An IndetermHyperRough Set further generalizes both IndetermRough Sets and HyperRough Sets by incorporating multiple attributes. It manages indeterminate relations and uncertain memberships across multi-attribute domains, utilizing rough approximations to model uncertainty more effectively. The following sections provide formal definitions and related theorems.

Definition 9 (IndetermRough Set). Let X be a non-empty universe of discourse. Let

$$R \subseteq X \times X$$

be an *indeterminate* relation on X , meaning that R may contain pairs whose membership in R is partially or fully unknown (or not well-defined). Equivalently or additionally, let there be a subset $U \subseteq X$ for which membership of certain elements $x \in X$ in U may also be indeterminate.

Formally, we say a triple

$$(X, R^*, U^*)$$

constitutes an *IndetermRough Set* if:

- (1) X is a finite or countably infinite set (the “universe”).
- (2) R^* is an equivalence-like relation on X with potential indeterminacy. That is, for each $x, y \in X$, we may have:

$$(x, y) \in R, \quad (x, y) \notin R, \quad \text{or } (x, y) \text{ is “indeterminate” in } R^*.$$

(In practical terms, we might know that x and y are equivalent, not equivalent, or we might *not* know which.)

- (3) U^* is a subset (or class of subsets) of X in which membership of an element x can be “determined” ($x \in U$ or $x \notin U$) or “indeterminate” (x ’s membership is unknown or not unique).

Next, define the *lower approximation* $\underline{U}^*$ and the *upper approximation* $\overline{U}^*$ with respect to R^* as follows:

$$\underline{U}^* = \{x \in X \mid \text{for every } y \text{ s.t. } (x, y) \text{ is definitely in } R, y \text{ is definitely in } U\},$$

$$\overline{U}^* = \{x \in X \mid \text{there exists some } y \text{ with } (x, y) \text{ possibly in } R^* \text{ and } y \text{ possibly in } U\}.$$

Here, “definitely in R ” means (x, y) is not indeterminate *and* is in R , while “possibly in R^* ” means (x, y) is either in R or its membership is unknown. Similarly, “definitely in U ” means y is not indeterminate in U , and “possibly in U ” means y is either in U or its membership is unknown.

The pair $(\underline{U}^*, \overline{U}^*)$ is then called the *indeterminate rough approximation* of U^* with respect to R^* . We often refer to the entire entity

$$(U^*, \underline{U}^*, \overline{U}^*)$$

as an *IndetermRough Set* over (X, R^*) .

Remark 10. In the above definition, the indeterminacy may appear in **any** of the following:

- The equivalence-like relation R^* itself.
- The subset U^* (uncertain membership for one or more elements).
- Both the relation R^* and the subset U^* simultaneously.

When no indeterminacy is present, R^* becomes a classical equivalence relation R , and U^* becomes a precise subset U . In that special case, IndetermRough Set theory reduces exactly to Pawlak’s classical Rough Set theory.

Example 11 (IndetermRough Set). Let $X = \{x_1, x_2, x_3\}$ be a small universe. Suppose we have a relation R^* that is mostly known but has one indeterminate pair:

$$R^* = \{(x_1, x_1), (x_1, x_2), (x_2, x_1), (x_2, x_2), (x_3, x_3)\} \cup \{\text{"possibly"} (x_2, x_3)\}.$$

Meaning we definitely know x_1 and x_2 are in the same equivalence class, but *we are not certain* whether x_2 and x_3 belong to the same class. Also let

$$U^* = \{(x_1, \text{definitely}), (x_2, \text{definitely}), (x_3, \text{unknown})\}.$$

In other words, we know $x_1, x_2 \in U^*$, but x_3 might or might not be in U^* .

Lower Approximation $\underline{U}^*$.

- $x_1 \in \underline{U}^*$ because $[x_1]_{R^*}$ (the definitely known part $\{x_1, x_2\}$) is contained in U^* for sure. (The pair (x_2, x_3) is only “possible,” so we do not definitely put x_3 in the same equivalence class as x_1 .)
- $x_2 \in \underline{U}^*$ by the same reasoning.
- $x_3 \notin \underline{U}^*$ because $[x_3]_{R^*}$ is at least $\{x_3\}$, and we do not know if $x_3 \in U^*$. So we *cannot* say $[x_3]_{R^*} \subseteq U^*$ definitely.

Hence $\underline{U}^* = \{x_1, x_2\}$.

Upper Approximation $\overline{U}^*$.

- $x_1 \in \overline{U}^*$ because $[x_1]_{R^*}$ definitely intersects U^* (at least x_1 or x_2).
- $x_2 \in \overline{U}^*$ for the same reason.
- $x_3 \in \overline{U}^*$ because if (x_2, x_3) is indeed in R (it is “possible”), then x_3 might join the equivalence class of x_2 , and $x_2 \in U^*$. So there is a *possible* intersection with U^* .

Hence $\overline{U}^* = \{x_1, x_2, x_3\}$.

Thus the IndetermRough Set

$$(U^*, \underline{U}^*, \overline{U}^*) = (\{x_1, x_2, ?\}, \{x_1, x_2\}, \{x_1, x_2, x_3\})$$

represents a situation where x_1 and x_2 are definitely in U^* , x_3 is indeterminate, and the equivalence between x_2 and x_3 is also not fully known.

Example 12 (IndetermRough Set in Job Interview Selection). Let

$$X = \{\text{Alice, Bob, Carol, Dave}\}$$

be the set of applicants. We define an *indeterminate* relation

$$R^* \subseteq X \times X$$

by

$$R^* = \{(\text{Alice, Alice}), (\text{Bob, Bob}), (\text{Carol, Carol}), (\text{Dave, Dave}), \\ (\text{Alice, Bob}), (\text{Bob, Alice})\} \cup \{(\text{possibly}) (\text{Bob, Carol}), (\text{Carol, Bob})\}.$$

Here Alice and Bob are definitely in the same screening group; it is uncertain whether Bob and Carol belong together.

Next, let

$$U^* = \{\text{Alice (definite), Carol (definite), Bob (unknown), Dave (unknown)}\}$$

be the (possibly indeterminate) subset of those *invited* for interview.

The indeterminate lower and upper approximations are then:

$$\underline{U}^* = \{x \in X \mid \forall y, (x, y) \text{ definite in } R^* \implies y \text{ definite in } U^*\}, \\ \overline{U}^* = \{x \in X \mid \exists y, (x, y) \text{ possibly in } R^* \wedge y \text{ possibly in } U^*\}.$$

A direct check yields:

$$\underline{U}^* = \{\text{Carol}\}, \quad \overline{U}^* = \{\text{Alice, Bob, Carol, Dave}\}.$$

Thus the IndetermRough Set

$$(U^*, \underline{U^*}, \overline{U^*})$$

captures that Carol is *definitely* invited (all her definite equivalents are in U^*), whereas all applicants are *possibly* invited.

Theorem 13 (IndetermRough Set Generalizes Rough Set). *Every classical Rough Set is a special case of an IndetermRough Set. Therefore, the class of IndetermRough Sets strictly includes the class of Rough Sets.*

Proof: We show that any classical Rough Set (X, R, U) can be realized as an IndetermRough Set with zero indeterminacy.

1. Recall the classical definitions.

Let X be a nonempty universe, $R \subseteq X \times X$ an equivalence relation, and $U \subseteq X$ a crisp subset. The classical lower and upper approximations of U are

$$\underline{U} = \{x \in X \mid [x]_R \subseteq U\}, \quad \overline{U} = \{x \in X \mid [x]_R \cap U \neq \emptyset\},$$

where $[x]_R = \{y \in X \mid (x, y) \in R\}$.

2. Construct the corresponding IndetermRough Set.

By Definition 9, an IndetermRough Set is given by a triple (X, R^*, U^*) in which

- R^* is an “equivalence-like” relation on X that may admit indeterminate pairs,
- $U^* \subseteq X$ is a (possibly indeterminate) target subset.

We now set

$$R^* = R \quad \text{and} \quad U^* = U,$$

interpreting “definitely in” and “possibly in” both simply as membership in the classical sense (i.e. there is no indeterminacy at all).

3. Verify the lower approximation agrees.

By the IndetermRough definition,

$$\underline{U^*} = \{x \in X \mid \forall y, (x, y) \text{ definite in } R^* \implies y \text{ definite in } U^*\}.$$

Since “definite in R^* ” means $(x, y) \in R$, and “definite in U^* ” means $y \in U$, this becomes

$$\underline{U^*} = \{x \in X \mid \forall y, (x, y) \in R \implies y \in U\} = \{x \in X \mid [x]_R \subseteq U\} = \underline{U}.$$

4. Verify the upper approximation agrees.

Similarly, the IndetermRough upper approximation is

$$\overline{U^*} = \{x \in X \mid \exists y, (x, y) \text{ possible in } R^* \wedge y \text{ possible in } U^*\}.$$

With no indeterminacy, “possible in R^* ” is again $(x, y) \in R$, and “possible in U^* ” is $y \in U$. Hence

$$\overline{U^*} = \{x \in X \mid \exists y \in X, (x, y) \in R \wedge y \in U\} = \{x \in X \mid [x]_R \cap U \neq \emptyset\} = \overline{U}.$$

5. Conclude the generalization.

Since the constructed IndetermRough Set (X, R^*, U^*) yields exactly the same approximations $\underline{U^*} = \underline{U}$ and $\overline{U^*} = \overline{U}$, every classical Rough Set appears as an IndetermRough Set with no indeterminacy. Thus

$$\{\text{Rough Sets}\} \subseteq \{\text{IndetermRough Sets}\}.$$

Finally, because IndetermRough Sets may allow genuine indeterminate pairs in R^* or uncertain membership in U^* , the inclusion is strict. This completes the proof. $\square$

Definition 14 (IndermHyperRough Set). Let X be a non-empty universe, and let $T_1, T_2, \dots, T_n$ be n distinct attributes (similar to the HyperRough Set case). Let each T_i have an attribute domain J_i , and set

$$J = J_1 \times J_2 \times \dots \times J_n.$$

Suppose further that for each $j \in J$, there is a subset $F^*(j) \subseteq X$ whose membership may be partially indeterminate, and an equivalence-like relation R^* on X that is also potentially indeterminate.

An *IndermHyperRough Set* is then the collection

$$(F^*, J),$$

where, for each $j \in J$:

- $F^*(j)$ is a (possibly indeterminate) subset of X .
- The *indeterminate lower approximation* of $F^*(j)$ is

$$\underline{F^*(j)} = \{x \in X \mid [x]_{R^*} \subseteq F^*(j) \text{ definitely, with } (x, y) \in R^* \text{ definitely}\}.$$

- The *indeterminate upper approximation* of $F^*(j)$ is

$$\overline{F^*(j)} = \{x \in X \mid [x]_{R^*} \cap F^*(j) \text{ is non-empty (possibly)}\}.$$

Above, $[x]_{R^*}$ denotes the (possibly indeterminate) equivalence class of x under R^* .

Hence each pair $(\underline{F^*(j)}, \overline{F^*(j)})$ captures the indeterminate rough approximations for the subset $F^*(j)$. We call

$$(\underline{F^*(j)}, \underline{F^*(j)}, \overline{F^*(j)})$$

an *indeterminate hyper-rough subset* for the attribute combination $j \in J$. The entire family

$$\{(\underline{F^*(j)}, \underline{F^*(j)}, \overline{F^*(j)}) \mid j \in J\}$$

is referred to as an *IndermHyperRough Set*.

Remark 15. Just as a HyperRough Set extends a Rough Set to multiple attributes (via a function $F(j)$ for j in the Cartesian product of attribute domains), an IndermHyperRough Set extends an IndermRough Set to multiple attributes and accounts for possible indeterminacy in each of:

- The equivalence-like relation R^* .
- The membership of each $F^*(j)$ in X .
- Both of the above.

In the absence of indeterminacy, we recover the standard HyperRough Set. If, in addition, we have only a single attribute domain, we recover the IndermRough Set from Definition 9.

Example 16 (IndermHyperRough Set). Let $X = \{x_1, x_2, x_3, x_4\}$ be a universe, and suppose there are two attributes T_1, T_2 with domains

$$J_1 = \{\alpha, \beta\}, \quad J_2 = \{\gamma, \delta\}.$$

Hence $J = J_1 \times J_2 = \{(\alpha, \gamma), (\alpha, \delta), (\beta, \gamma), (\beta, \delta)\}$.

Assume we have an indeterminate relation R^* on X , plus a mapping

$$F^* : J \rightarrow \mathcal{P}(X)$$

where each $F^*(j)$ may contain unknown memberships. For instance, let us define:

$$\begin{aligned} F^*(\alpha, \gamma) &= \{x_1, x_2, x_3 ?\}, \\ F^*(\alpha, \delta) &= \{x_1, x_2 ?\}, \\ F^*(\beta, \gamma) &= \{x_2, x_3, x_4 ?\}, \\ F^*(\beta, \delta) &= \{x_4 \text{ definitely}, x_1 ?\}. \end{aligned}$$

Here “ $x_3 ?$ ” means “ x_3 may or may not be in the set,” and so on.

To form the IndetermHyperRough Set, for each $j \in J$ we compute:

$$\underline{F^*(j)} \quad \text{and} \quad \overline{F^*(j)}$$

based on the (partially indeterminate) equivalence classes of X under R^* . For example, if R^* definitely equates x_1 and x_2 but is unsure about x_3 being in the same class, then x_3 might or might not belong to that class, etc. Consequently, each lower and upper approximation becomes partially uncertain.

All four pairs

$$\left(\underline{F^*(\alpha, \gamma)}, \overline{F^*(\alpha, \gamma)} \right), \quad \left(\underline{F^*(\alpha, \delta)}, \overline{F^*(\alpha, \delta)} \right), \quad \left(\underline{F^*(\beta, \gamma)}, \overline{F^*(\beta, \gamma)} \right), \quad \left(\underline{F^*(\beta, \delta)}, \overline{F^*(\beta, \delta)} \right),$$

collectively form the IndetermHyperRough Set (F^*, J) .

Example 17 (IndetermHyperRough Set in a Medical Diagnosis Scenario). Let

$$X = \{p_1, p_2, p_3, p_4, p_5\}$$

be a set of five patients. We consider two clinical attributes:

$$J_1 = \{\text{Mild, Moderate}\}, \quad J_2 = \{\text{Positive, Negative}\},$$

where “Mild/Moderate” refers to symptom severity and “Positive/Negative” to a laboratory marker result.

Define an indeterminate relation

$$R^* \subseteq X \times X$$

by

$$R^* = \{(p_1, p_1), (p_2, p_2), (p_3, p_3), (p_4, p_4), (p_5, p_5), \\ (p_1, p_2), (p_2, p_1), (p_3, p_4), (p_4, p_3)\} \cup \{(\text{possibly}) (p_2, p_3), (p_3, p_2)\}.$$

Here p_1 and p_2 are definitely in the same disease-subtype class, p_3 and p_4 likewise, and it is uncertain whether p_2 and p_3 share the same class.

Next, define

$$F^* : J_1 \times J_2 \longrightarrow \mathcal{P}(X)$$

by (“def” = definitely in, “?” = may or may not be in):

$$\begin{aligned} F^*(\text{Mild, Positive}) &= \{p_1 \text{ (def)}, p_2 \text{ (?)}, p_3 \text{ (?)}, p_5 \text{ (?)}\}, \\ F^*(\text{Mild, Negative}) &= \{p_2 \text{ (def)}, p_4 \text{ (?)}\}, \\ F^*(\text{Moderate, Positive}) &= \{p_3 \text{ (def)}, p_5 \text{ (?)}\}, \\ F^*(\text{Moderate, Negative}) &= \{p_4 \text{ (def)}, p_1 \text{ (?)}\}. \end{aligned}$$

We now compute the indeterminate lower and upper approximations for two cases:

$$\underline{F^*(\text{Mild, Positive})} = \{p_1\}, \quad \overline{F^*(\text{Mild, Positive})} = \{p_1, p_2, p_3, p_5\},$$

$$\underline{F^*(\text{Moderate, Negative})} = \{p_4\}, \quad \overline{F^*(\text{Moderate, Negative})} = \{p_1, p_4\}.$$

Similarly, one obtains $\underline{F^*(\text{Mild, Negative})} = \{p_2\}$, $\overline{F^*(\text{Mild, Negative})} = \{p_2, p_4\}$, $\underline{F^*(\text{Moderate, Positive})} = \{p_3\}$, and $\overline{F^*(\text{Moderate, Positive})} = \{p_3, p_5\}$.

Collecting all four pairs

$$(\underline{F^*(j)}, \overline{F^*(j)}) \quad \text{for } j \in J_1 \times J_2,$$

yields the IndetermHyperRough Set $(F^*, J_1 \times J_2)$.

Theorem 18 (IndetermHyperRough Set Generalizes IndetermRough and HyperRough). *The class of IndetermHyperRough Sets strictly includes:*

- (1) All IndetermRough Sets (by restricting to one attribute domain).

(2) *All HyperRough Sets (by removing all indeterminacy).*

Proof: We prove each inclusion separately, and then observe strictness.

(1) Every IndetermRough Set embeds into an IndetermHyperRough Set.

Let (X, R^*, U^*) be an arbitrary IndetermRough Set as in Definition 9. Thus

$$R^* \subseteq X \times X \quad (\text{possibly indeterminate equivalence-like relation}),$$

$$U^* \subseteq X \quad (\text{possibly indeterminate membership}).$$

Its indeterminate lower and upper approximations are

$$\underline{U^*} = \{x \in X \mid \forall y, (x, y) \text{ definite in } R^* \Rightarrow y \text{ definite in } U^*\},$$

$$\overline{U^*} = \{x \in X \mid \exists y, (x, y) \text{ possible in } R^* \wedge y \text{ possible in } U^*\}.$$

Now define an IndetermHyperRough Set as follows:

$$n = 1, \quad J_1 = \{\alpha\}, \quad J = J_1 = \{\alpha\},$$

$$F^* : J \rightarrow \mathcal{P}(X), \quad F^*(\alpha) = U^*,$$

and use the same relation R^* . Then by Definition 14 the approximations are

$$\underline{F^*(\alpha)} = \{x \in X \mid \forall y, (x, y) \text{ definite in } R^* \Rightarrow y \text{ definite in } F^*(\alpha)\},$$

$$\overline{F^*(\alpha)} = \{x \in X \mid \exists y, (x, y) \text{ possible in } R^* \wedge y \text{ possible in } F^*(\alpha)\}.$$

Since $F^*(\alpha) = U^*$, these coincide exactly with $\underline{U^*}$ and $\overline{U^*}$. Hence the IndetermHyperRough pair (F^*, J) reproduces the original IndetermRough Set. This shows:

$$\{\text{IndetermRough Sets}\} \subseteq \{\text{IndetermHyperRough Sets}\}.$$

(2) Every HyperRough Set embeds into an IndetermHyperRough Set.

Let (G, J) be a HyperRough Set on (X, R) (with no indeterminacy), where R is a classical equivalence relation and

$$G : J \rightarrow \mathcal{P}(X),$$

with lower and upper approximations

$$\underline{G(j)} = \{x \in X \mid [x]_R \subseteq G(j)\}, \quad \overline{G(j)} = \{x \in X \mid [x]_R \cap G(j) \neq \emptyset\}.$$

Define an IndetermHyperRough Set by

$$R^* = R \quad (\text{no indeterminacy}), \quad F^*(j) = G(j) \quad (\text{for each } j \in J).$$

Then “definite” and “possible” both coincide with membership in R or $G(j)$, so

$$\underline{F^*(j)} = \underline{G(j)}, \quad \overline{F^*(j)} = \overline{G(j)}.$$

Thus (G, J) is realized as an IndetermHyperRough Set with zero indeterminacy, proving

$$\{\text{HyperRough Sets}\} \subseteq \{\text{IndetermHyperRough Sets}\}.$$

(3) Strictness of Generalization.

To see that these inclusions are strict, observe that an IndetermHyperRough Set may have

- *Multiple* attributes ($n \geq 2$), so cannot be reduced to a single-attribute IndetermRough Set.
- *Non-trivial* indeterminacy in either the relation R^* or in some $F^*(j)$, so cannot be a pure HyperRough Set.

For example, take $n = 2$, $J_1 = J_2 = \{0, 1\}$, define a non-empty indeterminate subset $F^*(0, 0) \subset X$ and a “possibly” pair in R^* . This construction yields an IndermHyperRough Set not arising from either a one-attribute IndermRough or a fully determinate HyperRough. Hence

$$\{\text{IndermHyperRough Sets}\} \supsetneq \{\text{IndermRough Sets}\} \cup \{\text{HyperRough Sets}\},$$

completing the proof. $\square$

Theorem 19 (Monotonicity of Indeterminate Approximations). *Let (F^*, J) be an IndermHyperRough Set over (X, R^*) , and let $j \in J$. If $F_1^*(j) \subseteq F_2^*(j)$ (with the same relation R^* and the same pattern of indeterminacy), then*

$$\underline{F_1^*(j)} \subseteq \underline{F_2^*(j)}, \quad \overline{F_1^*(j)} \subseteq \overline{F_2^*(j)}.$$

Proof: We prove the two inclusions separately.

Lower approximation. Let $x \in \underline{F_1^*(j)}$. By definition, for every y such that (x, y) is *definitely* in R^* , y is *definitely* in $F_1^*(j)$. Since $F_1^*(j) \subseteq F_2^*(j)$, each such y is also *definitely* in $F_2^*(j)$. Therefore $x \in \underline{F_2^*(j)}$, showing

$$\underline{F_1^*(j)} \subseteq \underline{F_2^*(j)}.$$

Upper approximation. Let $x \in \overline{F_1^*(j)}$. Then there exists some y with (x, y) *possibly* in R^* and y *possibly* in $F_1^*(j)$. But $F_1^*(j) \subseteq F_2^*(j)$ implies y is also *possibly* in $F_2^*(j)$. Hence $x \in \overline{F_2^*(j)}$, proving

$$\overline{F_1^*(j)} \subseteq \overline{F_2^*(j)}.$$

$\square$

Theorem 20 (Union and Intersection Approximations). *Let (F^*, J) and (G^*, J) be two IndermHyperRough Sets over the same (X, R^*) . For each $j \in J$, define*

$$(F^* \cap G^*)(j) = F^*(j) \cap G^*(j), \quad (F^* \cup G^*)(j) = F^*(j) \cup G^*(j),$$

with indeterminacy inherited. Then:

$$\underline{(F^* \cap G^*)(j)} = \underline{F^*(j)} \cap \underline{G^*(j)}, \quad \overline{(F^* \cap G^*)(j)} = \overline{F^*(j)} \cap \overline{G^*(j)}.$$

Proof: We verify each identity.

Lower of intersection.

$$x \in \underline{(F^* \cap G^*)(j)} \iff \forall y, (x, y) \text{ definite in } R^* \Rightarrow y \in F^*(j) \cap G^*(j) \text{ definitely.}$$

This holds exactly when y is definitely in both $F^*(j)$ and $G^*(j)$ for all definite (x, y) , i.e.

$$x \in \underline{F^*(j)} \quad \text{and} \quad x \in \underline{G^*(j)},$$

which yields

$$\underline{(F^* \cap G^*)(j)} = \underline{F^*(j)} \cap \underline{G^*(j)}.$$

Upper of union.

$$x \in \overline{(F^* \cup G^*)(j)} \iff \exists y, (x, y) \text{ possible in } R^* \wedge y \in F^*(j) \cup G^*(j) \text{ possibly.}$$

Hence there is some y possibly in $F^*(j)$ or in $G^*(j)$, so

$$x \in \overline{F^*(j)} \quad \text{or} \quad x \in \overline{G^*(j)},$$

giving

$$\overline{(F^* \cup G^*)(j)} = \overline{F^*(j)} \cup \overline{G^*(j)}.$$

$\square$

Theorem 21 (Idempotence of Approximations). *For any IndetermHyperRough Set (F^*, J) over (X, R^*) and any $j \in J$, the operators $\underline{\cdot}$ and $\overline{\cdot}$ are idempotent:*

$$\underline{\underline{F^*(j)}} = \underline{F^*(j)}, \quad \overline{\overline{F^*(j)}} = \overline{F^*(j)}.$$

Proof: We show $\underline{\underline{F^*(j)}} = \underline{F^*(j)}$; the dual for the upper follows similarly.

Let $x \in \underline{\underline{F^*(j)}}$. By definition, for every y definitely in R^* , $y \in \underline{F^*(j)}$ definitely. But membership in $\underline{F^*(j)}$ already requires that for all definite $(y, z) \in R^*$, $z \in F^*(j)$ definitely. Chaining these two facts shows x satisfies the definition of $\underline{F^*(j)}$, so $\underline{\underline{F^*(j)}} \subseteq \underline{F^*(j)}$. The reverse inclusion $\underline{F^*(j)} \subseteq \underline{\underline{F^*(j)}}$ is immediate from monotonicity, since $\underline{F^*(j)} \subseteq F^*(j)$. Thus idempotence holds. $\square$

Theorem 22 (Boundary Region and Definability). *For (F^*, J) over (X, R^*) and $j \in J$, define the boundary region*

$$\text{BN}(F^*(j)) = \overline{F^*(j)} \setminus \underline{F^*(j)}.$$

Then $\text{BN}(F^*(j)) = \emptyset$ if and only if $F^*(j)$ is definable with respect to R^* (i.e. $F^*(j)$ is a union of definite equivalence classes and has no indeterminate membership).

Proof: $(\Rightarrow)$ If $\text{BN}(F^*(j)) = \emptyset$, then

$$\underline{F^*(j)} = \overline{F^*(j)}.$$

In classical Rough Set theory this equality characterizes precisely those subsets which are unions of whole equivalence classes. Here, since no element lies strictly between definite and possible membership, $F^*(j)$ must coincide with a union of definite classes without indeterminacy.

$(\Leftarrow)$ Conversely, if $F^*(j)$ is a union of some definite equivalence classes of R^* and each element's membership is completely known, then standard arguments show

$$\underline{F^*(j)} = F^*(j) = \overline{F^*(j)},$$

hence $\text{BN}(F^*(j)) = \emptyset$. $\square$

3 Conclusion and Future Work

This paper introduced and analysed two new mathematical frameworks—*IndetermRough Sets* and *IndetermHyperRough Sets*—that extend classical rough-set theory by embedding indeterminacy both in the underlying equivalence relations and in element membership.

Future research should undertake large-scale computational experiments and develop real-world applications for these extended models. We also foresee fruitful investigations into analogous indeterminacy-aware extensions built upon *Fuzzy Sets*[15, 16], *Intuitionistic Fuzzy Sets*[17, 18], *Neutrosophic Sets*[19, 20], *HyperFuzzy Sets*[21, 22], and *Hesitant Fuzzy Sets*[23], as well as graph-based counterparts expressed through *Graphs* [24], *Hypergraphs* [25, 26], and *SuperHypergraphs*[27, 28, 29].

Declarations

Ethics Approval and Consent to Participate

The results, data, and figures presented in this manuscript have not been published elsewhere, nor are they under consideration by another publisher. All the material is owned by the authors, and no permissions are required.

Consent for Publication

This article does not contain any studies involving human participants or animals performed by any of the authors.

Availability of Data and Materials

The data supporting the findings of this study are available from the corresponding author upon reasonable request.

Competing Interests

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