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HyperFuzzy Multisets and (m, n) -SuperHyperFuzzy Multisets: Case Studies and Applications in Medical Science and Diagnosis

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Abstract

Uncertainty in complex systems has motivated many well-known generalizations of classical set theory, including fuzzy sets, intuitionistic fuzzy sets, and other degree-based formalisms. In particular, a fuzzy set on a universe Y associates with each element $y \in Y$ a membership value in $[0, 1]$, so that belonging may vary continuously rather than being purely binary. Multisets extend ordinary sets by permitting repeated occurrences of the same element, and a fuzzy multiset refines this idea by allowing each element to be equipped with multiple membership values in $(0, 1]$. Hyperfuzzy sets broaden the model further by assigning to each $y \in Y$ a nonempty subset of $[0, 1]$, thereby capturing uncertainty together with intrinsic variability of membership. Continuing this line of development, (m, n) -superhyperfuzzy sets assign to each nonempty member arising from the m -fold and n -fold iterated powerset constructions a nonempty family of subsets of $[0, 1]$, which supports nested and hierarchical representations of imprecision.

A substantial body of work has also explored medical applications of fuzzy methods and their extensions, notably in medical diagnosis, where such tools have been widely employed.

Despite the broad use of fuzzy and hyperfuzzy notions in artificial intelligence, decision-making, algebra, topology, and control theory, more expressive structures—including hyperfuzzy multisets and (m, n) -superhyperfuzzy multisets—have received comparatively limited attention. In this paper we introduce these new objects, give precise formal definitions, and study their basic mathematical properties. We expect that subsequent research will design computational procedures and pursue application-oriented investigations that more fully exploit these advanced multiset frameworks. In addition, we provide several illustrative medical and biomedical examples of HyperFuzzy Multisets and SuperHyperFuzzy Multisets.

Keywords: Fuzzy Set, Fuzzy MultiSet, HyperFuzzy Set, MultiSet, HyperFuzzy MultiSet, (m, n) -Superhyperfuzzy Multiset



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1 | Preliminaries

We fix the basic notation and recall the foundational notions used throughout. Let Y be a nonempty universe. In the sense of Zadeh, a *fuzzy set* on Y is described by a membership map $\tau : Y \rightarrow [0, 1]$, where $\tau(y)$ represents the degree to which $y \in Y$ belongs to the set; this formalism captures graded membership rather than a crisp yes/no classification [1, 2, 3]. Closely related degree-based set models include intuitionistic fuzzy sets [4], neutrosophic sets [5, 6], and plithogenic sets [7].

We also require multiset terminology. For a set X , write $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. A (finite) multiset on X is encoded by a function $m : X \rightarrow \mathbb{N}_0$ with finite support, so that elements may occur with multiplicity greater than one (see, e.g., [8, 9]). A *fuzzy multiset* refines this idea by allowing each element of X to be associated with *multiple* membership values in $(0, 1]$ [10, 11]. In this paper we work with finite universes, hence all multisets and fuzzy multisets are finite.

Definition 1 (Fuzzy Set). [2, 3] Let Y be a nonempty set. A *fuzzy set* on Y is a mapping

$$\mu : Y \longrightarrow [0, 1],$$

where $\mu(y)$ is the membership degree of $y \in Y$. A *fuzzy relation* on Y is a function $\delta : Y \times Y \rightarrow [0, 1]$, i.e., a fuzzy subset of $Y \times Y$. We say that δ is a *fuzzy relation on μ* if, for all $y, z \in Y$,

$$\delta(y, z) \leq \min\{\mu(y), \mu(z)\}.$$

Definition 2 (Multiset). [12, 13] Let X be a set. A *finite multiset* on X is a function

$$m : X \longrightarrow \mathbb{N}_0$$

whose support

$$\text{supp}(m) := \{x \in X \mid m(x) > 0\}$$

is finite. We denote by $\mathcal{M}(X)$ the collection of all finite multisets on X . For each $x \in X$, the integer $m(x)$ is called the *multiplicity* of x in m .

Definition 3 (Fuzzy Multiset). [10, 11] Let X be a nonempty universe. Denote by $\text{MUS}(U)$ the class of all finite multisets on a set U . A *fuzzy multiset* A over X is a finite multiset of ordered pairs

$$A \subseteq X \times (0, 1],$$

together with its *counting function*

$$\text{Count}_A : X \longrightarrow \text{MUS}((0, 1]), \quad \text{Count}_A(x) = \{\{\mu \in (0, 1] \mid (x, \mu) \in A\}\},$$

so that $\text{Count}_A(x)$ is the finite multiset of membership values attached to x . Equivalently, A can be viewed as the multiset of all (x, μ) counted with their multiplicities:

$$A = \{(x, \mu) \mid \mu \in \text{Count}_A(x)\}.$$

For $x \in X$, suppose

$$\text{Count}_A(x) = \{\underbrace{\mu_{x,1}, \dots, \mu_{x,1}}_{m_{x,1} \text{ copies}}, \underbrace{\mu_{x,2}, \dots, \mu_{x,2}}_{m_{x,2} \text{ copies}}, \dots, \underbrace{\mu_{x,k}, \dots, \mu_{x,k}}_{m_{x,k} \text{ copies}}\},$$

where $\mu_{x,1} > \mu_{x,2} > \dots > \mu_{x,k} > 0$ and $m_{x,i} \in \mathbb{N}$. The corresponding *membership sequence* at x is the nonincreasing list

$$\underbrace{(\mu_{x,1}, \dots, \mu_{x,1})}_{m_{x,1} \text{ times}}, \underbrace{(\mu_{x,2}, \dots, \mu_{x,2})}_{m_{x,2} \text{ times}}, \dots, \underbrace{(\mu_{x,k}, \dots, \mu_{x,k})}_{m_{x,k} \text{ times}}.$$

We write $\text{FM}(X)$ for the class of all finite fuzzy multisets on X .

As supplementary information, Table 1 provides a concise comparison of a fuzzy set and a fuzzy multiset.

TABLE 1. Comparison of a fuzzy set and a fuzzy multiset.

Aspect	Fuzzy Set	Fuzzy Multiset
Universe	A nonempty set Y .	A nonempty set X (typically finite in this paper).
Basic data	A membership function $\mu : Y \rightarrow [0, 1]$.	A finite multiset $A \subseteq X \times (0, 1]$ (pairs (x, μ) counted with multiplicity).
Membership information per element	Each element $y \in Y$ carries exactly one degree $\mu(y)$.	Each element $x \in X$ may carry multiple degrees in $(0, 1]$, possibly repeated.
Counting / representation	No multiplicity notion; a single value represents the grade.	A counting function $\text{Count}_A : X \rightarrow \text{MUS}((0, 1])$ records all membership values (with multiplicities) attached to x .
Multiplicity	Not applicable.	Allowed both at the element level (repetition of (x, μ)) and at the degree level (repetition of the same μ for a fixed x).
Typical output type	Scalar grade in $[0, 1]$.	Finite multiset (or sequence) of grades in $(0, 1]$ for each element.
Relation to ordinary set/multiset	Reduces to a crisp set when $\mu(y) \in \{0, 1\}$ for all y .	Reduces to an ordinary multiset by ignoring degrees (or by fixing a single degree per occurrence).

1.1 | HyperFuzzy Sets and SuperHyperFuzzy Sets

Our discussion uses two higher-order membership formalisms. A *HyperFuzzy set* [14, 15] assigns to each element not a single number in $[0, 1]$ but a nonempty *set* of possible membership degrees. A further hierarchical extension is the *SuperHyperFuzzy set* [16, 17], where iterated powerset constructions are used to represent multi-level (nested) uncertainty. HyperFuzzy sets have also been investigated in algebra and related areas (e.g., [18, 19]).

Definition 4 (Underlying universe). A *base set* (or *universe*) is a fixed nonempty set S whose elements are the primary objects under study. All powerset and iterated-powerset constructions in this subsection are formed from S (or from related sets such as $[0, 1]$).

Definition 5 (Powerset). [20] For a set S , its *powerset* is

$$\text{POWER}(S) := \{A \mid A \subseteq S\},$$

that is, the family of all subsets of S , including $\emptyset$.

Definition 6 (HyperFuzzy set). [14] Let X be a nonempty set and let

$$\text{POWER}^{\sim}([0, 1]) := \{B \subseteq [0, 1] \mid B \neq \emptyset\}$$

denote the collection of all nonempty subsets of $[0, 1]$. A *HyperFuzzy set* on X is a mapping

$$\tilde{\mu} : X \longrightarrow \text{POWER}^{\sim}([0, 1]).$$

For each $x \in X$, the value $\tilde{\mu}(x) \subseteq [0, 1]$ is interpreted as the set of plausible membership degrees of x , capturing fuzziness together with internal variability.

Definition 7 (Iterated powersets and their nonempty versions). [21] Let H be any set. Define $\text{POWER}_1(H) := \text{POWER}(H)$ and, for $n \geq 1$,

$$\text{POWER}_{n+1}(H) := \text{POWER}(\text{POWER}_n(H)).$$

Define also the *nonempty iterated powersets* by removing $\emptyset$ at each stage:

$$\text{POWER}_1^*(H) := \text{POWER}(H) \setminus \{\emptyset\}, \quad \text{POWER}_{n+1}^*(H) := \text{POWER}(\text{POWER}_n^*(H)) \setminus \{\emptyset\} \quad (n \geq 1).$$

Definition 8 ((m, n) -SuperHyperFuzzy set). (cf. [17, 16, 22]) Let X be a nonempty set and let $m, n \in \mathbb{N}_0$. For any set Y , define the nonempty k -th powerset recursively by

$$\text{POWER}_0^*(Y) := Y, \quad \text{POWER}_k^*(Y) := \text{POWER}(\text{POWER}_{k-1}^*(Y)) \setminus \{\emptyset\} \quad (k \geq 1).$$

In particular, $\text{POWER}_m^*(X)$ consists of nonempty objects obtained after m layers of powerset formation on X , and $\text{POWER}_n^*([0, 1])$ is defined analogously.

Next, let

$$\text{POWER}_n^{\tilde{*}}([0, 1]) := \{ \Sigma \subseteq \text{POWER}_n([0, 1]) \mid \Sigma \neq \emptyset \},$$

i.e., the family of all nonempty subsets of $\text{POWER}_n([0, 1])$.

An (m, n) -SuperHyperFuzzy set on X is a function

$$\tilde{\mu}_{m,n} : \text{POWER}_m^*(X) \longrightarrow \text{POWER}_n^{\tilde{*}}([0, 1]), \quad A \longmapsto \tilde{\mu}_{m,n}(A).$$

Thus each $A \in \text{POWER}_m^*(X)$ is assigned a nonempty family $\tilde{\mu}_{m,n}(A) \subseteq \text{POWER}_n([0, 1])$ of n -level membership-degree sets, representing hierarchical uncertainty simultaneously at the m -level (objects in X grouped through m iterated powersets) and at the n -level (nested degree descriptions over $[0, 1]$).

Example 9 $((2, 2)$ -SuperHyperFuzzy set for two-stage control reliability). Let

$$X = \{\text{Sensor}, \text{Actuator}, \text{Controller}\}$$

represent the main components of a feedback control loop. Elements of $\text{POWER}_2^*(X)$ may encode two-stage subsystem groupings; for instance,

$$A_1 = \{\{\text{Sensor}, \text{Actuator}\}, \{\text{Controller}\}\}, \quad A_2 = \{\{\text{Sensor}\}, \{\text{Actuator}, \text{Controller}\}\}.$$

Define $\tilde{\mu}_{2,2} : \text{POWER}_2^*(X) \rightarrow \text{POWER}_2^{\tilde{*}}([0, 1])$ by

$$\tilde{\mu}_{2,2}(A_1) = \{\{0.80, 0.85\}, \{0.90\}\}, \quad \tilde{\mu}_{2,2}(A_2) = \{\{0.75, 0.80\}, \{0.88, 0.92\}\}.$$

Here each inner set (e.g., $\{0.80, 0.85\}$) is a plausible range of fuzzy reliability degrees for a given subsystem grouping, while the outer braces list alternative scenarios considered by the engineer. Hence uncertainty is represented both in the grouped object A_i ($m = 2$) and in the degree-description layer ($n = 2$).

Example 10 (A real-world $(1, 2)$ -SuperHyperFuzzy set). Let

$$X = \{\text{Model A}, \text{Model B}\}$$

be two smartphone models and take $(m, n) = (1, 2)$. Then

$$\text{POWER}_1^*(X) = \{\{\text{Model A}\}, \{\text{Model B}\}, \{\text{Model A}, \text{Model B}\}\}.$$

Moreover,

$$\text{POWER}_2([0, 1]) = \text{POWER}(\text{POWER}([0, 1])), \quad \text{POWER}_2^{\tilde{*}}([0, 1]) = \{ S \subseteq \text{POWER}_2([0, 1]) \mid S \neq \emptyset \}.$$

Define $\tilde{\mu}_{1,2} : \text{POWER}_1^*(X) \rightarrow \text{POWER}_2^{\tilde{*}}([0, 1])$ as follows:

- For $\{\text{Model A}\}$, durability is evaluated in two environments (laboratory vs. field), each producing a nested cluster of plausible degree-sets:

$$\tilde{\mu}_{1,2}(\{\text{Model A}\}) = \{ \{ \{0.90, 0.92\}, \{0.88, 0.91\} \}, \{ \{0.85, 0.87\}, \{0.83, 0.86\} \} \}.$$

- For $\{\text{Model B}\}$, only field tests are available, yielding one nested cluster:

$$\tilde{\mu}_{1,2}(\{\text{Model B}\}) = \{ \{ \{0.75, 0.78\}, \{0.72, 0.76\} \} \}.$$

- For $\{\text{Model A}, \text{Model B}\}$, a user survey provides a single nested estimate:

$$\tilde{\mu}_{1,2}(\{\text{Model A}, \text{Model B}\}) = \{ \{ \{0.80, 0.85\} \} \}.$$

Accordingly, $\tilde{\mu}_{1,2}$ assigns to each nonempty subset of models a nonempty family of second-order membership-degree sets, representing both multiple evidence sources and measurement variability in a hierarchical manner.

For reference, Table 2 presents a compact overview of fuzzy sets, HyperFuzzy sets, and (m, n) -SuperHyperFuzzy sets.

TABLE 2. Compact overview of fuzzy sets, HyperFuzzy sets, and (m, n) -SuperHyperFuzzy sets.

Model	Membership assignment	What it captures (informal)
Fuzzy set	$\mu : X \rightarrow [0, 1]$	Each element $x \in X$ has a single degree $\mu(x)$; uncertainty is represented only as one scalar value.
HyperFuzzy set	$\tilde{\mu} : X \rightarrow \text{POWER}^{\sim}([0, 1])$	Each $x \in X$ is assigned a <i>nonempty set</i> $\tilde{\mu}(x) \subseteq [0, 1]$ of plausible degrees, so variability in the degree itself is modeled explicitly.
(m, n) -SuperHyperFuzzy set	$\tilde{\mu}_{m,n} : \text{POWER}_m^*(X) \rightarrow \text{POWER}_n^{\sim}([0, 1])$	Domain objects are m -level (nonempty) iterated-powerset objects $A \in \text{POWER}_m^*(X)$, and each such A receives a nonempty family $\tilde{\mu}_{m,n}(A) \subseteq \text{POWER}_n([0, 1])$ of n -level degree-descriptions; this encodes nested (multi-level) uncertainty on both the object side (m) and the degree side (n).

2 | Main Results

This section contains the central contributions of the paper. We introduce two multiset-based extensions of higher-order fuzzy membership: *HyperFuzzy multisets* and (m, n) -*SuperHyperFuzzy multisets*. Both formalisms provide a systematic way to encode (i) multiple membership assessments, (ii) uncertainty expressed by sets of plausible degrees, and (iii) hierarchical/nested imprecision. Such features are particularly relevant for medical information, where measurements and diagnoses often exhibit variability, partial evidence, and overlapping interpretations.

2.1 | HyperFuzzy Multiset

Informally, a HyperFuzzy multiset on X assigns to each element $x \in X$ a *multiset* whose entries are nonempty subsets of $[0, 1]$. Hence one may store several contextual evaluations for the same element, and each evaluation may itself be set-valued.

Definition 11 (HyperFuzzy multiset). Let X be a nonempty set. A *HyperFuzzy multiset* on X is a mapping

$$H : X \longrightarrow \text{MUS}(\text{POWER}^{\sim}([0, 1]))$$

such that the support

$$\text{Supp}(H) := \{x \in X \mid H(x) \neq \text{MUS}(\emptyset)\}$$

is finite. The class of all HyperFuzzy multisets on X is denoted by $\text{HyFMS}(X)$.

Remark 12. An equivalent viewpoint is to treat a HyperFuzzy multiset as a finite multiset of pairs

$$H \subseteq X \times \text{POWER}^{\sim}([0, 1]),$$

where (x, S) may occur with a finite multiplicity. The associated counting function records, for each $x \in X$, the multiset of set-valued degrees:

$$\text{Count}_H(x) := \{(S \mid (x, S) \in H)\}, \quad S \subseteq [0, 1], S \neq \emptyset.$$

Example 13 (HyperFuzzy multiset from clustered product ratings). Consider a product-review platform where scores range from 0 (poor) to 1 (excellent). Because ratings depend on rater groups and usage contexts, it is natural to represent each context by a *set* of plausible degrees. Let

$$X = \{\text{Product A}, \text{Product B}\}.$$

Define $H : X \rightarrow \text{MUS}(\text{POWER}([0, 1]))$ by

$$\begin{aligned} H(\text{Product A}) &= \{(\{0.90, 0.92\}), (\{0.88, 0.91\}), (\{0.85, 0.89\})\}, \\ H(\text{Product B}) &= \{(\{0.75, 0.78\}), (\{0.70, 0.73\})\}. \end{aligned}$$

Each inner set $\{a, b\} \subseteq [0, 1]$ represents a context-dependent range of acceptable membership degrees (e.g., laboratory test vs. field trial vs. consumer survey), while the multiset collects multiple contexts. Since $\text{Supp}(H) = \{\text{Product A}, \text{Product B}\}$ is finite, we have $H \in \text{HyFMS}(X)$.

Example 14 (HyperFuzzy multiset for comfort-forecast variability). Let Station_1 and Station_2 be two local weather stations evaluating the fuzzy concept “comfortable afternoon temperature.” Suppose each station uses three forecasting sources (e.g., NOAA, ECMWF, and a regional ensemble). For each source one obtains a nonempty set of plausible membership degrees in $[0, 1]$. Put

$$X = \{\text{Station}_1, \text{Station}_2\}$$

and define $H : X \rightarrow \text{MUS}(\text{POWER}([0, 1]))$ by

$$\begin{aligned} H(\text{Station}_1) &= \{(\{0.80, 0.85\}), (\{0.75, 0.90\}), (\{0.82, 0.88\})\}, \\ H(\text{Station}_2) &= \{(\{0.60, 0.65\}), (\{0.55, 0.70\}), (\{0.62, 0.68\})\}. \end{aligned}$$

Thus $H \in \text{HyFMS}(X)$ and simultaneously encodes multiplicity of sources and uncertainty within each source.

Theorem 15 (Embedding fuzzy multisets and hyperfuzzy sets). *The class $\text{HyFMS}(X)$ contains faithful copies of both fuzzy multisets $\text{FM}(X)$ and HyperFuzzy sets $\text{POWER}([0, 1])^X$. More precisely:*

(i) *There exists an injective map $\Phi : \text{FM}(X) \rightarrow \text{HyFMS}(X)$ given by*

$$\Phi(A)(x) := \{\{\mu\} \mid \mu \in \text{Count}_A(x)\},$$

i.e., each scalar membership value μ is encoded as the singleton set $\{\mu\}$.

(ii) *There exists an injective map $\Psi : \text{POWER}([0, 1])^X \rightarrow \text{HyFMS}(X)$ given by*

$$\Psi(f)(x) := \{f(x)\},$$

i.e., each set-valued membership $f(x)$ is stored as a single multiset entry.

Proof: (i) Let $A \in \text{FM}(X)$. For each $x \in X$, $\text{Count}_A(x)$ is a finite multiset of degrees in $(0, 1]$. Define $\Phi(A)(x)$ by replacing every $\mu \in \text{Count}_A(x)$ with the singleton $\{\mu\}$. Then $\Phi(A)(x) \in \text{MUS}(\text{POWER}([0, 1]))$ and the support remains finite, so $\Phi(A) \in \text{HyFMS}(X)$. If $A \neq A'$, then some x has $\text{Count}_A(x) \neq \text{Count}_{A'}(x)$, hence $\Phi(A)(x) \neq \Phi(A')(x)$; therefore Φ is injective.

(ii) Let $f \in \text{POWER}([0, 1])^X$. Since $f(x) \neq \emptyset$ for all x , the singleton multiset $\{f(x)\}$ is well-defined and yields $\Psi(f) \in \text{HyFMS}(X)$ with finite support. If $f \neq g$, then $f(x) \neq g(x)$ for some x , hence $\Psi(f)(x) \neq \Psi(g)(x)$; thus Ψ is injective. $\square$

Theorem 16 (Stability under multiset union and intersection). *Let $H_1, H_2 \in \text{HyFMS}(X)$. For each $x \in X$, define*

$$(H_1 \uplus H_2)(x) := H_1(x) \uplus H_2(x), \quad (H_1 \cap_M H_2)(x) := H_1(x) \cap_M H_2(x),$$

where $\uplus$ is multiset union and $\cap_M$ is multiset intersection (minimum of multiplicities). Then $H_1 \uplus H_2 \in \text{HyFMS}(X)$ and $H_1 \cap_M H_2 \in \text{HyFMS}(X)$.

Proof: Because $\text{Supp}(H_1)$ and $\text{Supp}(H_2)$ are finite, both supports are contained in $\text{Supp}(H_1) \cup \text{Supp}(H_2)$, hence finite. For each x , the multiset union and multiset intersection of two finite multisets in $\text{MUS}(\text{POWER}([0, 1]))$ are again finite multisets in the same codomain. Therefore both $H_1 \uplus H_2$ and $H_1 \cap_M H_2$ satisfy Definition 11. $\square$

Theorem 17 (Flattening to a HyperFuzzy set). *Define*

$$\text{Flat} : \text{HyFMS}(X) \longrightarrow \text{POWER}^{\sim}([0, 1])^X, \quad \text{Flat}(H)(x) := \bigcup_{S \in H(x)} S,$$

where the union ranges over the (set-valued) entries of the multiset $H(x)$. Then $\text{Flat}(H)$ is a HyperFuzzy set on X , and Flat is surjective.

Proof: Fix $x \in X$. Since $H(x)$ is a finite multiset of nonempty subsets of $[0, 1]$, at least one entry S exists whenever $x \in \text{Supp}(H)$, and each such S satisfies $S \neq \emptyset$. Hence the union $\bigcup_{S \in H(x)} S$ is a nonempty subset of $[0, 1]$. Therefore $\text{Flat}(H) \in \text{POWER}^{\sim}([0, 1])^X$.

To prove surjectivity, let $f : X \rightarrow \text{POWER}^{\sim}([0, 1])$ be arbitrary. Define $H \in \text{HyFMS}(X)$ by

$$H(x) := \{(f(x))\} \quad (x \in X).$$

Then $\text{Flat}(H)(x) = f(x)$ for every x , so Flat is onto. $\square$

Theorem 18 (Lower and upper envelopes). *Let $H \in \text{HyFMS}(X)$. Define functions $\underline{\mu}_H, \bar{\mu}_H : X \rightarrow [0, 1]$ by*

$$\underline{\mu}_H(x) := \min\{\min S \mid S \in H(x)\}, \quad \bar{\mu}_H(x) := \max\{\max S \mid S \in H(x)\}.$$

Then $\underline{\mu}_H$ and $\bar{\mu}_H$ are fuzzy sets on X .

Proof: For any x , each entry $S \in H(x)$ is a nonempty subset of $[0, 1]$, hence $\min S$ and $\max S$ are well-defined elements of $[0, 1]$. Because $H(x)$ has finitely many entries, the finite collections $\{\min S \mid S \in H(x)\}$ and $\{\max S \mid S \in H(x)\}$ admit a minimum and a maximum in $[0, 1]$. Thus $\underline{\mu}_H(x), \bar{\mu}_H(x) \in [0, 1]$ for all x , so both are fuzzy sets. $\square$

Definition 19 (Complement). Let $H \in \text{HyFMS}(X)$. The *complement* H^c is the HyperFuzzy multiset defined by

$$H^c(x) := \{([0, 1] \setminus S) \mid S \in H(x)\}, \quad x \in X,$$

i.e., each set-valued entry is complemented inside $[0, 1]$ while multiplicities are preserved.

Theorem 20 (Complement is an involution). *For every $H \in \text{HyFMS}(X)$, one has $(H^c)^c = H$.*

Proof: For any $S \subseteq [0, 1]$,

$$[0, 1] \setminus ([0, 1] \setminus S) = S.$$

Applying Definition 19 twice complements every entry twice and does not change multiplicities. Hence $(H^c)^c(x) = H(x)$ for all $x \in X$, so $(H^c)^c = H$. $\square$

Theorem 21 (Restriction to a subuniverse). *Let $H \in \text{HyFMS}(X)$ and let $A \subseteq X$. Define*

$$H|_A : A \longrightarrow \text{MUS}(\text{POWER}^{\sim}([0, 1])), \quad H|_A(x) := H(x) \quad (x \in A).$$

Then $H|_A \in \text{HyFMS}(A)$.

Proof: The support of $H|_A$ equals $\text{Supp}(H) \cap A$, which is finite because $\text{Supp}(H)$ is finite. For each $x \in A$, the value $H|_A(x) = H(x)$ is a finite multiset of nonempty subsets of $[0, 1]$. Therefore $H|_A$ satisfies Definition 11 on the universe A . $\square$

2.2 | SuperHyperFuzzy Multiset

An (m, n) -SuperHyperFuzzy multiset associates to each nonempty m -level object built from X a multiset of nonempty n -level membership-value objects, thereby encoding multivalued and hierarchically nested uncertainty.

Definition 22 (Nested-singleton operator). Let Y be a nonempty set. For each integer $k \geq 0$ define a map $\iota_k^Y : Y \rightarrow \text{POWER}_k^*(Y)$ by

$$\iota_0^Y(y) = y, \quad \iota_{k+1}^Y(y) = \{\iota_k^Y(y)\}.$$

Then $\iota_k^Y(y) \in \text{POWER}_k^*(Y)$ holds for every $y \in Y$. In particular, $\iota_m^X(x) \in \text{POWER}_m^*(X)$ for $x \in X$, and $\iota_n^{[0,1]}(\mu) \in \text{POWER}_n^*([0, 1])$ for $\mu \in [0, 1]$.

Definition 23 ((m, n) -SuperHyperFuzzy multiset). Let X be a nonempty set and let $m, n \in \mathbb{N}_0$. An (m, n) -SuperHyperFuzzy multiset on X is a mapping

$$H : \text{POWER}_m^*(X) \longrightarrow \text{MUS}(\text{POWER}_n^*([0, 1]))$$

whose *support* is finite, i.e.,

$$\text{Supp}(H) := \{A \in \text{POWER}_m^*(X) \mid H(A) \neq \text{MUS}(\emptyset)\} \text{ is finite.}$$

The family of all such objects is denoted by $\text{SuHyFMS}_{m,n}(X)$.

Example 24 $((1, 2)$ -SuperHyperFuzzy multiset: hotel satisfaction under repeated surveys). Let $X = \{\text{Hotel A}, \text{Hotel B}\}$ and $(m, n) = (1, 2)$. Then

$$\text{POWER}_1^*(X) = \{\{\text{Hotel A}\}, \{\text{Hotel B}\}, \{\text{Hotel A}, \text{Hotel B}\}\}.$$

Define

$$H : \text{POWER}_1^*(X) \longrightarrow \text{MUS}(\text{POWER}_2^*([0, 1]))$$

by

$$\begin{aligned} H(\{\text{Hotel A}\}) &= \{(\{\{0.90, 0.92\}, \{0.88, 0.91\}\}, \{\{0.85, 0.87\}, \{0.83, 0.86\}\}, \{\{0.90, 0.92\}, \{0.88, 0.91\}\})\}, \\ H(\{\text{Hotel B}\}) &= \{(\{\{0.75, 0.78\}, \{0.72, 0.76\}\}, \{\{0.75, 0.78\}, \{0.72, 0.76\}\})\}, \\ H(\{\text{Hotel A}, \text{Hotel B}\}) &= \{(\{\{0.80, 0.85\}\})\}. \end{aligned}$$

Each innermost pair $\{a, b\} \subseteq [0, 1]$ describes a plausible range of “satisfaction” degrees obtained from one survey method; the next level groups ranges into a second-order cluster; and the outer multiset counts repeated survey rounds (so duplicates indicate repetition). Since

$$\text{Supp}(H) = \{\{\text{Hotel A}\}, \{\text{Hotel B}\}, \{\text{Hotel A}, \text{Hotel B}\}\}$$

is finite, we have $H \in \text{SuHyFMS}_{1,2}(X)$. Thus H stores hierarchical uncertainty across hotel subsets together with multiplicity across survey rounds.

Example 25 $((2, 3)$ -SuperHyperFuzzy multiset: products, survey rounds, and nested variability). Let

$$X = \{\text{Product A}, \text{Product B}\}, \quad (m, n) = (2, 3).$$

For brevity write $A = \text{Product A}$ and $B = \text{Product B}$. Then

$$\begin{aligned} \text{POWER}_1^*(X) &= \{\{A\}, \{B\}, \{A, B\}\}, \\ \text{POWER}_2^*(X) &= \{\{\{A\}\}, \{\{B\}\}, \{\{A, B\}\}, \{\{A\}, \{B\}\}\}. \end{aligned}$$

Define

$$H : \text{POWER}_2^*(X) \longrightarrow \text{MUS}(\text{POWER}_3^*([0, 1]))$$

by setting

$$H(\{\{A\}, \{B\}\}) = \{(T_1, T_2)\}, \quad H(\{\{A, B\}\}) = \{(U)\},$$

and $H(C) = \text{MUS}(\emptyset)$ for all remaining $C \in \text{POWER}_2^*(X)$, where

$$\begin{aligned} T_1 &= \{(\{\{0.90, 0.92\}, \{0.88, 0.91\}\}, \{\{0.85, 0.87\}, \{0.83, 0.86\}\})\}, \\ T_2 &= \{(\{\{0.78, 0.80\}, \{0.76, 0.79\}\}, \{\{0.82, 0.84\}, \{0.80, 0.83\}\})\}, \\ U &= \{(\{\{0.82, 0.85\}\}, \{\{0.80, 0.83\}\})\}. \end{aligned}$$

Here each innermost $\{a, b\} \subseteq [0, 1]$ is a cluster of plausible degrees from one survey; the next layer groups clusters by survey round; and the outer multiset accounts for repeated survey campaigns. Since

$$\text{Supp}(H) = \{\{\{A\}, \{B\}\}, \{\{A, B\}\}\}$$

is finite, $H \in \text{SuHyFMS}_{2,3}(X)$. Hence H realizes three nested levels of uncertainty together with multiplicity.

Theorem 26 (Containment of $\text{FM}(X)$, $\text{HyFMS}(X)$, and (m, n) -SHFS). *The class $\text{SuHyFMS}_{m,n}(X)$ properly extends each of the following:*

$$\text{FM}(X), \quad \text{HyFMS}(X), \quad (\text{POWER}_n^*([0, 1]))^{\text{POWER}_m^*(X)} \quad (\text{the class of } (m, n)\text{-SuperHyperFuzzy sets}).$$

More explicitly:

(i) *There exists an injective map $\Phi : \text{FM}(X) \rightarrow \text{SuHyFMS}_{m,n}(X)$ defined by*

$$\Phi(A)(B) = \begin{cases} \{(\iota_n^{[0,1]}(\mu)) \mid \mu \in \text{Count}_A(x)\}, & \text{if } B = \iota_m^X(x) \text{ for some } x \in X, \\ \text{MUS}(\emptyset), & \text{otherwise.} \end{cases}$$

(ii) *There exists an injective map $\Psi : \text{HyFMS}(X) \rightarrow \text{SuHyFMS}_{m,n}(X)$ defined by*

$$\Psi(H_0)(B) = \begin{cases} \{S' \mid S \in \text{Count}_{H_0}(x)\}, & \text{if } B = \iota_m^X(x) \text{ and } S' = \iota_n^{[0,1]}(S), \\ \text{MUS}(\emptyset), & \text{otherwise.} \end{cases}$$

(iii) *There exists an injective map*

$$\Theta : (\text{POWER}_n^*([0, 1]))^{\text{POWER}_m^*(X)} \longrightarrow \text{SuHyFMS}_{m,n}(X)$$

given by

$$\Theta(f)(B) = \begin{cases} \{(f(B))\}, & B \in \text{POWER}_m^*(X), \\ \text{MUS}(\emptyset), & \text{otherwise.} \end{cases}$$

Proof: (i) Fix $A \in \text{FM}(X)$. For each $x \in X$, the multiset $\text{Count}_A(x)$ is finite. The definition of $\Phi(A)$ places nonempty values only on indices $\iota_m^X(x)$ with $x \in \text{Supp}(A)$, hence

$$\text{Supp}(\Phi(A)) \subseteq \{\iota_m^X(x) \mid x \in \text{Supp}(A)\},$$

which is finite. Therefore $\Phi(A) \in \text{SuHyFMS}_{m,n}(X)$. If $A \neq A'$, then for some x , $\text{Count}_A(x) \neq \text{Count}_{A'}(x)$, hence $\Phi(A)(\iota_m^X(x)) \neq \Phi(A')(\iota_m^X(x))$, so Φ is injective.

(ii) Let $H_0 \in \text{HyFMS}(X)$. Because $\text{Supp}(H_0)$ is finite, $\Psi(H_0)$ is nonempty only on $\iota_m^X(x)$ with $x \in \text{Supp}(H_0)$, so its support is finite and $\Psi(H_0) \in \text{SuHyFMS}_{m,n}(X)$. Distinct H_0 yield distinct $\text{Count}_{H_0}(x)$ for some x , hence distinct images; thus Ψ is injective.

(iii) Let $f : \text{POWER}_m^*(X) \rightarrow \text{POWER}_n^*([0, 1])$. For each index $B \in \text{POWER}_m^*(X)$, $\Theta(f)(B)$ is the singleton multiset containing $f(B)$, so the mapping is well-defined. If $f \neq g$, then $f(B) \neq g(B)$ for some B , hence $\Theta(f)(B) \neq \Theta(g)(B)$, proving injectivity.

Consequently, $\text{SuHyFMS}_{m,n}(X)$ contains injective images of the three stated classes, and therefore properly generalizes them. $\square$

Theorem 27 (Closure under multiset union). *Let $H_1, H_2 \in \text{SuHyFMS}_{m,n}(X)$. Define a mapping $H_1 \cup H_2$ by*

$$(H_1 \cup H_2)(A) := H_1(A) \uplus H_2(A) \quad (A \in \text{POWER}_m^*(X)),$$

where $\uplus$ denotes multiset union (addition of multiplicities). Then $H_1 \cup H_2 \in \text{SuHyFMS}_{m,n}(X)$.

Proof: Since $\text{Supp}(H_1)$ and $\text{Supp}(H_2)$ are finite,

$$\text{Supp}(H_1 \cup H_2) = \text{Supp}(H_1) \cup \text{Supp}(H_2)$$

is finite. For each A , the multiset $H_1(A) \uplus H_2(A)$ lies in the same codomain $\text{MUS}(\text{POWER}_n^*([0, 1]))$ and remains finite. Hence $H_1 \cup H_2$ satisfies Definition 23. $\square$

Theorem 28 (Intersection decomposition at nested-singleton indices). *Let $H \in \text{SuHyFMS}_{m,n}(X)$ and let $x, y \in X$. Assume that*

$$\iota_m^X(x) \cap \iota_m^X(y) \in \text{Supp}(H).$$

Then

$$H(\iota_m^X(x) \cap \iota_m^X(y)) = H(\iota_m^X(x)) \cap_M H(\iota_m^X(y)),$$

where $\cap_M$ denotes multiset intersection (minimum of multiplicities).

Proof: At depth m , the nested-singleton construction yields

$$\iota_m^X(x) \cap \iota_m^X(y) = \iota_m^X(z) \iff x = y = z.$$

Under the stated assumption, the index $\iota_m^X(x) \cap \iota_m^X(y)$ is a support element of H , and the multiset intersection $\cap_M$ is well-defined in the codomain and preserves finiteness. The common entries are exactly those appearing in both multisets with multiplicity given by the minimum; hence the displayed equality holds. $\square$

Theorem 29 (Embedding via singleton multisets). *Let $\delta : \text{POWER}_n^*([0, 1]) \rightarrow \text{MUS}(\text{POWER}_n^*([0, 1]))$ be the map sending each element to the corresponding singleton multiset. Then for any fixed (m, n) -SuperHyperFuzzy set $\tilde{\mu} : \text{POWER}_m^*(X) \rightarrow \text{POWER}_n^*([0, 1])$, the composite*

$$X \xrightarrow{\iota_m^X} \text{POWER}_m^*(X) \xrightarrow{\tilde{\mu}} \text{POWER}_n^*([0, 1]) \xrightarrow{\delta} \text{MUS}(\text{POWER}_n^*([0, 1]))$$

induces an injection $X \rightarrow \text{SuHyFMS}_{m,n}(X)$, embedding each $x \in X$ as a “pure” (m, n) -SuperHyperFuzzy multiset.

Proof: For each $x \in X$, Definition 22 gives $\iota_m^X(x) \in \text{POWER}_m^*(X)$. Applying $\tilde{\mu}$ yields a nonempty element of $\text{POWER}_n^*([0, 1])$, and then δ produces the singleton multiset of that value. Distinct x give distinct indices $\iota_m^X(x)$, hence distinct images, proving injectivity. Each resulting mapping has support $\{\iota_m^X(x)\}$, which is finite, so it lies in $\text{SuHyFMS}_{m,n}(X)$. $\square$

Theorem 30 (Distributive lattice structure). *With operations $\uplus$ (multiset union) and $\cap_M$ (multiset intersection), the class $\text{SuHyFMS}_{m,n}(X)$ forms a distributive lattice. Concretely, for all $H_1, H_2, H_3 \in \text{SuHyFMS}_{m,n}(X)$ the following identities hold:*

- (1) $H \uplus H = H$ and $H \cap_M H = H$,
- (2) $H_1 \uplus H_2 = H_2 \uplus H_1$ and $H_1 \cap_M H_2 = H_2 \cap_M H_1$,
- (3) $(H_1 \uplus H_2) \uplus H_3 = H_1 \uplus (H_2 \uplus H_3)$ and similarly for $\cap_M$,
- (4) $H_1 \uplus (H_1 \cap_M H_2) = H_1$ and $H_1 \cap_M (H_1 \uplus H_2) = H_1$,
- (5) $H_1 \uplus (H_2 \cap_M H_3) = (H_1 \uplus H_2) \cap_M (H_1 \uplus H_3)$.

Proof: All statements reduce pointwise to identities of multiplicities in $\mathbb{N}_0$, because both $\uplus$ and $\cap_M$ act by, respectively, addition and minimum of multiplicities. In particular, associativity and commutativity come from the corresponding properties of $+$ and $\min$, absorption follows from $\max(a, \min(a, b)) = a$ and $\min(a, \max(a, b)) = a$ (with $\max$ interpreted via multiset union), and distributivity follows from

$$\max(a, \min(b, c)) = \min(\max(a, b), \max(a, c)) \quad (a, b, c \in \mathbb{N}_0).$$

Since $\text{SuHyFMS}_{m,n}(X)$ is closed under these pointwise multiset operations, it inherits the stated lattice structure. $\square$

Theorem 31 (Flattening to HyperFuzzy multisets). *Define*

$$\text{Flat} : \text{SuHyFMS}_{m,n}(X) \longrightarrow \text{HyFMS}(X), \quad \text{Flat}(H)(x) := H(\iota_m^X(x)), \quad x \in X.$$

Then $\text{Flat}(H) \in \text{HyFMS}(X)$ for every $H \in \text{SuHyFMS}_{m,n}(X)$, and Flat is surjective.

Proof: For each $x \in X$, $\iota_m^X(x) \in \text{POWER}_m^*(X)$, hence $\text{Flat}(H)(x)$ lies in $\text{MUS}(\text{POWER}_n^*([0,1])) \subseteq \text{MUS}(\text{POWER}([0,1]))$. Moreover, if $\text{Flat}(H)(x) \neq \text{MUS}(\emptyset)$ then $\iota_m^X(x) \in \text{Supp}(H)$, so

$$\text{Supp}(\text{Flat}(H)) \subseteq \{x \in X \mid \iota_m^X(x) \in \text{Supp}(H)\}$$

is finite. Hence $\text{Flat}(H) \in \text{HyFMS}(X)$.

For surjectivity, let $G \in \text{HyFMS}(X)$ be given. Define $H \in \text{SuHyFMS}_{m,n}(X)$ by

$$H(A) = \begin{cases} G(x), & \text{if } A = \iota_m^X(x) \text{ for some } x \in X, \\ \text{MUS}(\emptyset), & \text{otherwise.} \end{cases}$$

Then $\text{Flat}(H)(x) = H(\iota_m^X(x)) = G(x)$ for all x , so Flat is onto. $\square$

Theorem 32 (Level-augmentation injection). *Let $p \geq n$. There exists an injective map*

$$\text{Aug}_{n \rightarrow p} : \text{SuHyFMS}_{m,n}(X) \longrightarrow \text{SuHyFMS}_{m,p}(X)$$

defined by nesting membership-value sets by $p - n$ additional singleton layers. Concretely, for $H \in \text{SuHyFMS}_{m,n}(X)$ and $A \in \text{POWER}_m^*(X)$, the multiset $\text{Aug}_{n \rightarrow p}(H)(A)$ is obtained from $H(A)$ by replacing every entry $S \in \text{POWER}_n^*([0,1])$ with $\iota_{p-n}^{[0,1]}(S) \in \text{POWER}_p^*([0,1])$, preserving multiplicities.

Proof: By Definition 22, if $S \in \text{POWER}_n^*([0,1])$ then $\iota_{p-n}^{[0,1]}(S) \in \text{POWER}_p^*([0,1])$, so the construction lands in the correct codomain. Supports are unchanged because entries are only re-encoded, so finiteness of $\text{Supp}(H)$ implies finiteness of $\text{Supp}(\text{Aug}_{n \rightarrow p}(H))$.

If $\text{Aug}_{n \rightarrow p}(H_1) = \text{Aug}_{n \rightarrow p}(H_2)$, then for each A the resulting multisets coincide. Since the transformation $S \mapsto \iota_{p-n}^{[0,1]}(S)$ is injective, the original multisets $H_1(A)$ and $H_2(A)$ must coincide for all A , hence $H_1 = H_2$. Therefore $\text{Aug}_{n \rightarrow p}$ is injective. $\square$

3 | Additional Result

As additional results of this paper, we present the notions of Fuzzy Multi-Offset and Uncertain Multiset.

3.1 | Fuzzy Multi-Offset

A number of generalizations of fuzzy sets have been developed. Among them, the notions of fuzzy offset, fuzzy overset, and fuzzy underset are particularly notable [23, 24]. Closely related ideas, including neutrosophic offsets and their operator-based variants, have also been investigated [25, 26]. Motivated by this line of research, we study a multiset-valued counterpart, called a *Fuzzy Multi-Offset*, and give a precise definition together with basic relationships to existing classes.

Definition 33 (Fuzzy Offset, Overset, and Underset). [24] Let X be a universe of discourse and fix real numbers $\Psi < 0 < 1 < \Omega$. A *generalized fuzzy set* on X is a mapping

$$\mu : X \longrightarrow [\Psi, \Omega].$$

According to whether μ takes values beyond the standard interval $[0, 1]$, we distinguish:

- (1) *Fuzzy overset*: $\mu(x) \leq \Omega$ for all $x \in X$ with $\Omega > 1$, and there exists $x_0 \in X$ such that $\mu(x_0) > 1$.
- (2) *Fuzzy underset*: $\Psi < \mu(x) \leq 1$ for all $x \in X$ with $\Psi < 0$, and there exists $y_0 \in X$ such that $\mu(y_0) < 0$.
- (3) *Fuzzy offset*: $\mu(x) \in [\Psi, \Omega]$ for all $x \in X$ with $\Psi < 0 < 1 < \Omega$, and there exist $x_1, y_1 \in X$ such that $\mu(x_1) > 1$ and $\mu(y_1) < 0$.

Definition 34 (Fuzzy Multi-Offset). Let X be a nonempty set and fix real numbers $\Psi < 0 < 1 < \Omega$. A *Fuzzy Multi-Offset* on X is a mapping

$$M : X \longrightarrow \text{MUS}([\Psi, \Omega])$$

such that

$$\text{Supp}(M) = \{x \in X \mid M(x) \neq \text{MUS}(\emptyset)\}$$

is finite, and there exist $x_1, x_2 \in X$ satisfying

$$\exists \mu > 1 : \mu \in M(x_1) \quad \text{and} \quad \exists \nu < 0 : \nu \in M(x_2).$$

Definition 35 (Fuzzy Multi-Overset). With the same notation, a *Fuzzy Multi-Overset* on X is a mapping

$$M^+ : X \longrightarrow \text{MUS}([0, \Omega])$$

with finite support such that there exist $x_0 \in X$ and $\mu > 1$ with $\mu \in M^+(x_0)$.

Definition 36 (Fuzzy Multi-Underset). With the same notation, a *Fuzzy Multi-Underset* on X is a mapping

$$M^- : X \longrightarrow \text{MUS}([\Psi, 1])$$

with finite support such that there exist $y_0 \in X$ and $\nu < 0$ with $\nu \in M^-(y_0)$.

Example 37 (Engine-temperature monitoring as a Fuzzy Multi-Offset). Consider an automotive engine-temperature control setting equipped with two sensors:

$$X = \{\text{Sensor}_1, \text{Sensor}_2\}, \quad \Psi = -0.2, \quad \Omega = 1.3.$$

Define a mapping $M : X \rightarrow \text{MUS}([\Psi, \Omega])$ by collecting membership degrees produced by three diagnostic channels (internal thermistor, infrared imaging, and a software-based model):

$$M(\text{Sensor}_1) = \{(-0.10, 0.45, 0.75, 1.20)\}, \quad M(\text{Sensor}_2) = \{(0.20, 0.55, 1.05)\}.$$

Interpretation:

- $-0.10 < 0$ (for Sensor_1) corresponds to an “undershoot” signal, indicating an apparent undercooling reading below the nominal safe range.
- $1.20 > 1$ (for Sensor_1) and $1.05 > 1$ (for Sensor_2) correspond to “overshoot” signals, indicating overheating beyond the expected maximum.
- Values in $(0, 1)$ represent ordinary fuzzy membership degrees within the nominal range.

Since $\text{Supp}(M) = \{\text{Sensor}_1, \text{Sensor}_2\}$ is finite and M contains both a value below 0 and a value above 1, the mapping M satisfies Definition 34. Thus M is a Fuzzy Multi-Offset that aggregates under- and overtemperature indications from multiple diagnostic sources in a single multiset-valued representation.

Theorem 38 (Generalization of fuzzy multisets and offset-classes). Let X be a nonempty set and fix $\Psi < 0 < 1 < \Omega$.

- (1) There is an injective mapping

$$\Phi : \text{FM}(X) \longrightarrow \{\text{Fuzzy Multi-Offsets on } X\}, \quad \Phi(A)(x) = \{(\mu) \mid \mu \in \text{Count}_A(x)\}, \quad x \in X.$$

- (2) There is an injective mapping

$$\Psi_{\text{off}} : \{\text{Fuzzy Offsets on } X\} \longrightarrow \{\text{Fuzzy Multi-Offsets on } X\}, \quad \Psi_{\text{off}}(\mu_{\bar{A}})(x) = \{(\mu_{\bar{A}}(x))\}, \quad x \in X.$$

- (3) By the same pattern, one obtains injective embeddings of $\text{FM}(X)$ and the class of fuzzy oversets into the class of Fuzzy Multi-Oversets, and of $\text{FM}(X)$ and the class of fuzzy undersets into the class of Fuzzy Multi-Undersets.

Consequently, each of the three multi-classes properly extends both the fuzzy-multiset class and the corresponding offset-class.

Proof: (i) Let $A \in \text{FM}(X)$. For each $x \in X$, $\text{Count}_A(x)$ is a finite multiset of membership values in $(0, 1] \subseteq [\Psi, \Omega]$. Hence $\Phi(A)(x)$ is a finite multiset in $[\Psi, \Omega]$ and

$$\text{Supp}(\Phi(A)) = \text{Supp}(A)$$

is finite. If $A \neq A'$, then $\text{Count}_A(x) \neq \text{Count}_{A'}(x)$ for some x , implying $\Phi(A)(x) \neq \Phi(A')(x)$. Thus Φ is injective.

(ii) Let $\mu_{\tilde{A}} : X \rightarrow [\Psi, \Omega]$ be a fuzzy offset in the sense of Definition 33. Define $\Psi_{\text{off}}(\mu_{\tilde{A}})(x)$ as the singleton multiset containing $\mu_{\tilde{A}}(x)$. Then the support is

$$\text{Supp}(\Psi_{\text{off}}(\mu_{\tilde{A}})) = \{x \in X \mid \mu_{\tilde{A}}(x) \neq 0\},$$

and the offset condition (existence of values > 1 and < 0) transfers immediately to the multiset values. Distinct functions $\mu_{\tilde{A}}$ produce distinct singleton multisets, so Ψ_{off} is injective.

(iii) The overset and underset cases are obtained by the identical constructions, replacing the codomain interval by $[0, \Omega]$ or $[\Psi, 1]$ and imposing only the overshoot or undershoot existence condition. The same finiteness and injectivity arguments apply.

Therefore, each multi-class contains faithful images of both $\text{FM}(X)$ and the corresponding offset-class, proving the claimed generalization. $\square$

For reference, Table 3 presents a compact overview of fuzzy multisets and their multi-offset variants.

TABLE 3. Compact overview of fuzzy multisets and their multi-offset variants.

Model	Membership assignment	Defining condition(s)
Fuzzy Multiset $\text{FM}(X)$	$\text{Count}_A : X \rightarrow \text{MUS}((0, 1])$	Finite support $\text{Supp}(A) = \{x \in X \mid \text{Count}_A(x) \neq \text{MUS}(\emptyset)\}$; multiplicities encode repeated degrees in $(0, 1]$.
Fuzzy Multi-Offset	$M : X \rightarrow \text{MUS}([\Psi, \Omega]) \quad (\Psi < 0 < 1 < \Omega)$	Finite support; there exist $x_1, x_2 \in X$ with $\exists \mu > 1 : \mu \in M(x_1)$ and $\exists \nu < 0 : \nu \in M(x_2)$.
Fuzzy Multi-Overset	$M^+ : X \rightarrow \text{MUS}([0, \Omega]) \quad (\Omega > 1)$	Finite support; there exist $x_0 \in X$ and $\mu > 1$ such that $\mu \in M^+(x_0)$.
Fuzzy Multi-Underset	$M^- : X \rightarrow \text{MUS}([\Psi, 1]) \quad (\Psi < 0)$	Finite support; there exist $y_0 \in X$ and $\nu < 0$ such that $\nu \in M^-(y_0)$.

3.2 | Uncertain Multisets

As an additional result, we introduce *Uncertain Multisets*. Conceptually, this notion is obtained by replacing the single uncertainty value assigned by a U -set with a finite multiset of uncertainty values, thereby allowing repeated and multi-source assessments for the same element. We keep the same general philosophy of U -sets: the nature of uncertainty is controlled by a chosen degree-domain $\text{Dom}(M)$ [27, 28].

Definition 39 (Uncertain Set (U-Set)). [28] Let X be a nonempty set and let M be an uncertainty model with degree-domain

$$\text{Dom}(M) \subseteq [0, 1]^k$$

for some integer $k \geq 1$. A U -Set of type M on X is a mapping

$$\mu_M : X \longrightarrow \text{Dom}(M).$$

For each $x \in X$, the element $\mu_M(x) \in \text{Dom}(M)$ is called the M -membership degree (or M -uncertainty value) of x . With suitable choices of M , this definition recovers the standard degree-based set models (fuzzy, intuitionistic fuzzy, neutrosophic, plithogenic, and others).

Definition 40 (Uncertain Multiset (U-Multiset) of type M). Let X be a nonempty set and let M be an uncertainty model with degree-domain

$$\text{Dom}(M) \subseteq [0, 1]^k$$

for some integer $k \geq 1$. An *Uncertain Multiset of type M* on X is a mapping

$$U_M : X \longrightarrow \text{MUS}(\text{Dom}(M))$$

such that its support

$$\text{Supp}(U_M) = \{x \in X \mid U_M(x) \neq \text{MUS}(\emptyset)\}$$

is finite. For each $x \in X$, the multiset $U_M(x) \in \text{MUS}(\text{Dom}(M))$ is called the *M -uncertainty multiset* (or *multivalued M -membership*) of x . We write $\text{UMS}_M(X)$ for the class of all uncertain multisets of type M on X .

Equivalently, one may view U_M as a finite multiset of pairs

$$U_M \subseteq X \times \text{Dom}(M),$$

where the multiplicity of (x, d) records how many times the uncertainty value $d \in \text{Dom}(M)$ is assigned to $x \in X$.

Theorem 41 (Uncertain multisets generalize U -sets). Let X be a nonempty set and let M be an uncertainty model with degree-domain $\text{Dom}(M) \subseteq [0, 1]^k$. Then there exists an injective mapping

$$\iota : \{U\text{-sets of type } M \text{ on } X\} \longrightarrow \text{UMS}_M(X)$$

defined by

$$(\iota(\mu_M))(x) = \{(\mu_M(x))\}, \quad x \in X,$$

that is, $\iota(\mu_M)(x)$ is the singleton multiset containing the value $\mu_M(x)$. Consequently, $\text{UMS}_M(X)$ is a strict extension of the class of U -sets of type M .

Proof: Let $\mu_M : X \rightarrow \text{Dom}(M)$ be a U -set of type M . Define $U_M = \iota(\mu_M)$ by $U_M(x) = \{\mu_M(x)\}$ for each $x \in X$. Then $U_M(x) \in \text{MUS}(\text{Dom}(M))$ holds for all x , and $\text{Supp}(U_M) \subseteq X$ is finite whenever the underlying context assumes finite support, so $U_M \in \text{UMS}_M(X)$ by Definition 40. (In particular, if X is finite, then $\text{Supp}(U_M) = X$ is finite.)

To prove injectivity, suppose $\iota(\mu_M) = \iota(\nu_M)$. Then for every $x \in X$ we have

$$\{(\mu_M(x))\} = \{(\nu_M(x))\},$$

which forces $\mu_M(x) = \nu_M(x)$. Hence $\mu_M = \nu_M$. Therefore ι is injective, and $\text{UMS}_M(X)$ contains a faithful copy of the U -sets of type M . $\square$

Theorem 42 (Uncertain multisets generalize fuzzy multisets). Let X be a nonempty set. Fix the uncertainty model M_F with

$$\text{Dom}(M_F) = [0, 1] \subseteq [0, 1]^1.$$

Then the class $\text{UMS}_{M_F}(X)$ contains an isomorphic copy of the class of fuzzy multisets $\text{FM}(X)$. More precisely, there exists an injective map

$$\Phi : \text{FM}(X) \longrightarrow \text{UMS}_{M_F}(X)$$

such that, for each $A \in \text{FM}(X)$ and $x \in X$,

$$\Phi(A)(x) = \text{Count}_A(x) \in \text{MUS}([0, 1]).$$

In particular, $\text{UMS}_{M_F}(X)$ strictly generalizes fuzzy multisets.

Proof: Let $A \in \text{FM}(X)$. By definition of a fuzzy multiset, $\text{Count}_A(x)$ is a finite multiset of values in $(0, 1] \subseteq [0, 1]$ for each $x \in X$, and the set

$$\text{Supp}(A) = \{x \in X \mid \text{Count}_A(x) \neq \text{MUS}(\emptyset)\}$$

is finite. Define $\Phi(A) : X \rightarrow \text{MUS}([0, 1])$ by $\Phi(A)(x) = \text{Count}_A(x)$. Then $\Phi(A) \in \text{UMS}_{M_F}(X)$ because $\text{Dom}(M_F) = [0, 1]$ and $\text{Supp}(\Phi(A)) = \text{Supp}(A)$ is finite.

To show injectivity, assume $\Phi(A) = \Phi(B)$ for $A, B \in \text{FM}(X)$. Then for all $x \in X$,

$$\text{Count}_A(x) = \Phi(A)(x) = \Phi(B)(x) = \text{Count}_B(x).$$

A fuzzy multiset is determined uniquely by its counting function $x \mapsto \text{Count}(x)$, hence $A = B$. Therefore Φ is injective, and $\text{FM}(X)$ embeds into $\text{UMS}_{M_F}(X)$. $\square$

4 | Case Study: Applications in Medicine and Medical Science

Fuzzy- and neutrosophic-based formalisms have long been used in medical settings, including medical-diagnosis tasks (see, e.g., [29, 30]). Motivated by this established line of work, we present in this section several illustrative application examples of HyperFuzzy Multisets and SuperHyperFuzzy Multisets in Medicine and Medical Science. We also expect that subsequent studies will complement these theoretical formulations with computational experiments and related validation methodologies.

Example 43 (Antibiotic Clinical Success in Community-Acquired Pneumonia). Let

$$X = \{\text{Amoxicillin}, \text{Levofloxacin}\}.$$

Consider the fuzzy notion “*clinical success at day 14*,” whose degrees lie in $[0, 1]$. Because different evidence sources (randomized trials, observational cohorts, meta-analytic summaries) can produce *interval-like* assessments rather than a single value, we model the evidence as ranges of plausible degrees. Define a HyperFuzzy Multiset $H : X \rightarrow \text{MUS}(\text{POWER}([0, 1]))$ by

$$H(\text{Amoxicillin}) = \{\{0.78, 0.82\}, \{0.74, 0.80\}, \{0.76, 0.84\}\},$$

$$H(\text{Levofloxacin}) = \{\{0.83, 0.88\}, \{0.79, 0.86\}\}.$$

Here each inner nonempty set $S \subseteq [0, 1]$ represents an evidence-derived range, while the outer braces collect these ranges as a multiset (allowing multiple contexts).

Lower and upper envelopes. For $x \in X$,

$$\underline{\mu}_H(x) = \min\{\min S \mid S \in H(x)\}, \quad \bar{\mu}_H(x) = \max\{\max S \mid S \in H(x)\}.$$

Therefore,

$$\underline{\mu}_H(\text{Amoxicillin}) = \min\{0.78, 0.74, 0.76\} = 0.74, \quad \bar{\mu}_H(\text{Amoxicillin}) = \max\{0.82, 0.80, 0.84\} = 0.84,$$

$$\underline{\mu}_H(\text{Levofloxacin}) = \min\{0.83, 0.79\} = 0.79, \quad \bar{\mu}_H(\text{Levofloxacin}) = \max\{0.88, 0.86\} = 0.88.$$

Hence, under conservative aggregation, the admissible success band is $[0.74, 0.84]$ for Amoxicillin and $[0.79, 0.88]$ for Levofloxacin.

Example 44 (Lesion Malignancy Suspicion from Multi-Modal Imaging). Let

$$X = \{\text{Lesion A}, \text{Lesion B}\}.$$

For the fuzzy concept “*malignancy suspicion*,” three imaging modalities (MRI, CT, and Ultrasound) provide ranges of plausible degrees. Define $H : X \rightarrow \text{MUS}(\text{POWER}([0, 1]))$ by

$$H(\text{Lesion A}) = \{\{0.78, 0.82\}, \{0.70, 0.76\}, \{0.65, 0.72\}\},$$

$$H(\text{Lesion B}) = \{\{0.55, 0.63\}, \{0.60, 0.68\}, \{0.58, 0.66\}\}.$$

Envelopes. For Lesion A,

$$\underline{\mu}_H(\text{A}) = \min\{0.78, 0.70, 0.65\} = 0.65, \quad \bar{\mu}_H(\text{A}) = \max\{0.82, 0.76, 0.72\} = 0.82.$$

For Lesion B,

$$\underline{\mu}_H(\text{B}) = \min\{0.55, 0.60, 0.58\} = 0.55, \quad \bar{\mu}_H(\text{B}) = \max\{0.63, 0.68, 0.66\} = 0.68.$$

Thus a conservative suspicion band is $[0.65, 0.82]$ for Lesion A and $[0.55, 0.68]$ for Lesion B.

Example 45 $((m, n) = (1, 2)$: Dose Adjustment Necessity in Special Populations). Let

$$X = \{\text{Elderly } (E), \text{Renal Impairment } (R)\}.$$

We model the fuzzy notion “*dose adjustment necessity*.” The choice $m = 1$ means that the domain indices are nonempty subsets of X , while $n = 2$ means that membership values are *sets of subsets* of $[0, 1]$. Hence $H : \text{POWER}_1^*(X) \rightarrow \text{MUS}(\text{POWER}_2^*([0, 1]))$.

The nonempty subsets of X are

$$\{E\}, \quad \{R\}, \quad \{E, R\}.$$

We assign to each subset A a multiset of “scenario clusters” T , where each cluster is a finite family of subsets of $[0, 1]$ obtained from distinct protocols:

$$\begin{aligned} H(\{E\}) &= \{T_1^{(E)}, T_2^{(E)}\}, \quad T_1^{(E)} = \{\{0.70, 0.78\}, \{0.74, 0.82\}\}, \quad T_2^{(E)} = \{\{0.65, 0.73\}, \{0.72, 0.80\}\}, \\ H(\{R\}) &= \{T_1^{(R)}, T_1^{(R)}\}, \quad T_1^{(R)} = \{\{0.80, 0.88\}, \{0.76, 0.84\}\}, \\ H(\{E, R\}) &= \{T_1^{(E,R)}\}, \quad T_1^{(E,R)} = \{\{0.85, 0.92\}, \{0.83, 0.90\}\}. \end{aligned}$$

Repeated occurrences (e.g. two copies of $T_1^{(R)}$) indicate duplicated or corroborating evidence. Conservative envelope on $\text{POWER}_1^*(X)$. For any $A \subseteq X$ with $A \neq \emptyset$, define

$$\underline{\mu}_H(A) = \min_{T \in H(A)} \min_{S \in T} S, \quad \bar{\mu}_H(A) = \max_{T \in H(A)} \max_{S \in T} S.$$

Then

$$\begin{aligned} \underline{\mu}_H(\{E\}) &= \min\{0.70, 0.74, 0.65, 0.72\} = 0.65, \quad \bar{\mu}_H(\{E\}) = \max\{0.78, 0.82, 0.73, 0.80\} = 0.82, \\ \underline{\mu}_H(\{R\}) &= \min\{0.80, 0.76\} = 0.76, \quad \bar{\mu}_H(\{R\}) = \max\{0.88, 0.84\} = 0.88, \\ \underline{\mu}_H(\{E, R\}) &= \min\{0.85, 0.83\} = 0.83, \quad \bar{\mu}_H(\{E, R\}) = \max\{0.92, 0.90\} = 0.92. \end{aligned}$$

Therefore the combined subgroup $\{E, R\}$ exhibits the strongest dose-adjustment necessity within the band $[0.83, 0.92]$.

Example 46 $((m, n) = (2, 2)$: Sepsis Bundle Configuration and Response). Let the intervention set be

$$X = \{\text{Antibiotic } (A), \text{Fluids } (F), \text{Vasopressor } (V)\}.$$

With $m = 2$, elements of $\text{POWER}_2^*(X)$ are *collections of nonempty subsets* of X , expressing a two-stage grouping. Consider

$$\text{PRB}_1 = \{\{A, F\}, \{V\}\}, \quad \text{PRB}_2 = \{\{A\}, \{F, V\}\},$$

which represent two clinically plausible bundle organizations in early sepsis care. We evaluate the fuzzy concept “*timely hemodynamic stabilization*.”

With $n = 2$, membership values are scenario clusters T , each being a finite family of $[0, 1]$ -subsets. Define $H : \text{POWER}_2^*(X) \rightarrow \text{MUS}(\text{POWER}_2^*([0, 1]))$ by

$$\begin{aligned} H(\text{PRB}_1) &= \{T_1^{(1)}, T_2^{(1)}\}, \quad T_1^{(1)} = \{\{0.82, 0.88\}, \{0.90\}\}, \quad T_2^{(1)} = \{\{0.78, 0.84\}, \{0.86, 0.91\}\}, \\ H(\text{PRB}_2) &= \{T_1^{(2)}\}, \quad T_1^{(2)} = \{\{0.76, 0.83\}, \{0.80, 0.85\}\}. \end{aligned}$$

(A singleton such as $\{0.90\}$ indicates a particularly tight estimate from high-certainty evidence.)

Conservative envelope on $\text{POWER}_2^*(X)$. For $\text{PRB} \in \{\text{PRB}_1, \text{PRB}_2\}$, define

$$\underline{\mu}_H(\text{PRB}) = \min_{T \in H(\text{PRB})} \min_{S \in T} S, \quad \bar{\mu}_H(\text{PRB}) = \max_{T \in H(\text{PRB})} \max_{S \in T} S.$$

Then

$$\begin{aligned} \underline{\mu}_H(\text{PRB}_1) &= \min\{0.82, 0.90, 0.78, 0.86\} = 0.78, \quad \bar{\mu}_H(\text{PRB}_1) = \max\{0.88, 0.90, 0.84, 0.91\} = 0.91, \\ \underline{\mu}_H(\text{PRB}_2) &= \min\{0.76, 0.80\} = 0.76, \quad \bar{\mu}_H(\text{PRB}_2) = \max\{0.83, 0.85\} = 0.85. \end{aligned}$$

Thus, under these inputs, PRB_1 yields a stronger stabilization band $[0.78, 0.91]$ than PRB_2 with $[0.76, 0.85]$.

5 | Conclusion

In this paper, we introduced the notions of HyperFuzzy multisets and (m, n) -SuperHyperFuzzy multisets, presented their formal definitions, and studied their key properties. These structures capture multiplicity, context variability, and hierarchical uncertainty in a unified manner, enabling robust envelopes, flexible aggregation, and stronger decision support in multi-layer systems. And beyond their theoretical significance, these structures hold promise for diagnostic applications, where robust modeling of uncertainty can enhance patient classification, support clinical decision-making, and improve the reliability of automated medical systems. As a result of this study, we provide in Table 4 an overview comparing Multisets, Fuzzy Multisets, HyperFuzzy Multisets, and (m, n) -SuperHyperFuzzy Multisets.

TABLE 4. Overview of Multisets, Fuzzy Multisets, HyperFuzzy Multisets, and (m, n) -SuperHyperFuzzy Multisets

Object	Carrier / Domain	Mapping (formal)	Codomain	Concrete numeric example
Multiset	Finite base set X	Multiplicity $m : X \rightarrow \mathbb{N}_0$; $\text{Supp}(m) = \{x \mid m(x) > 0\}$	$\mathbb{N}_0$	$X = \{a, b\}$, $m(a) = 2$, $m(b) = 1$. Bag: $\{\{a, a, b\}\}$.
Fuzzy Multiset	Finite X	Finite multiset $A \subseteq X \times (0, 1]$; $\text{Count}_A : X \rightarrow \text{MUS}((0, 1])$	Multiset of degrees $(0, 1]$	$X = \{p, q\}$, $A = \{(p, 0.90), (p, 0.70), (q, 0.50)\}$. $\text{Count}_A(p) = \{0.90, 0.70\}$.
HyperFuzzy Multiset	Finite X	$H : X \rightarrow \text{MUS}(\text{POWER}([0, 1]))$	Multiset of nonempty degree-sets	$X = \{p, q\}$. $H(p) = \{\{0.70, 0.80\}, \{0.90\}\}$, $H(q) = \{\{0.55, 0.65\}\}$. $\text{Flat}(H)(p) = \{0.70, 0.80, 0.90\}$; envelope $[0.70, 0.90]$.
(m, n) -SuperHyperFuzzy Multiset	Finite X ; $m, n \in \mathbb{N}_0$	$H : \text{POWER}_m^*(X) \rightarrow \text{MUS}(\text{POWER}_n^*([0, 1]))$	Multiset of n th-level degree-sets	$X = \{u, v\}$, $m=1$, $n=2$. $H(\{u\}) = \{\{\{0.84, 0.88\}, \{0.80, 0.85\}\}\}$; $H(\{u, v\}) = \{\{\{0.72, 0.78\}\}\}$.

Notes: $\text{POWER}(S) = \text{nonempty subsets of } S$; $\text{POWER}_0^*(X) = X$, $\text{POWER}_{k+1}^*(X) = \text{POWER}(\text{POWER}_k^*(X)) \setminus \{\emptyset\}$. Canonical inclusions: $\text{FM}(X) \hookrightarrow \text{HyFMS}(X) \hookrightarrow \text{SuHyFMS}_{m,n}(X)$.

Regarding future directions, we anticipate developing extensions grounded in Neutrosophic sets, Plithogenic sets, Intuitionistic Fuzzy sets, Picture Fuzzy sets, Pythagorean Fuzzy sets, and Hesitant Fuzzy sets, and exploring applications in decision-making, machine learning, and algorithm design. Moreover, because this paper conducts only theoretical investigations, we hope that subsequent work will carry out quantitative analyses—such as computational experiments and studies using datasets—to validate the suitability and effectiveness of the models proposed here.

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Data Availability

The datasets generated and/or analyzed during the current study are not publicly available due to privacy-preserving constraints but are available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare that there is no conflict of interest in this research.

Ethical Approval

This article does not contain any studies involving human participants or animals conducted by any of the authors.

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