

**Paper Type: Original Article**

Hierarchical Network Modeling with Intuitionistic Fuzzy Superhypergraphs

Takaaki Fujita¹ , Muhammad Saqlain² , and Muhammad Gulistan^{3*} ,

¹ Independent Researcher, Tokyo, Japan; Takaaki.fujita060@gmail.com.

² School of Computer Science, Faculty of Engineering and Information Technology, University of Technology Sydney, 81 Broadway, Ultimo 2007, NSW, Australia. Muhamamd.Saqlain@uts.edu.au

^{3*} Department of Mathematics and Statistics, Hazara University Mansehra, Mansehra, Pakistan. gulistanm21@yahoo.com.

Received: 20 Aug 2025

Revised: 03 Dec 2025

Accepted: 14 Jan 2026

Published: 16 Jan 2026

Abstract

Traditional graphs capture only pairwise relationships, whereas hypergraphs allow edges to connect arbitrary subsets of vertices, modeling higher-order interactions. Superhypergraphs extend this idea further by iteratively applying the powerset operation to reveal nested, hierarchical connections. Concurrently, fuzzy-logic theories—such as fuzzy sets and intuitionistic fuzzy sets—provide graded notions of membership and uncertainty. Despite the individual success of these frameworks, the combination of hierarchical superhypergraph structures with intuitionistic fuzziness has received little attention. In this paper, we introduce the *intuitionistic fuzzy superhypergraph*, in which each supervertex and superedge carries both membership and non-membership degrees subject to Atanassov's constraint. We establish its formal properties, present an efficient construction algorithm, and demonstrate how it can underpin a novel decision-making procedure for complex, uncertain, multi-scale networks.

Keywords: Superhypergraph, Hypergraph, Fuzzy Graph, Intuitionistic Fuzzy Graph, Intuitionistic Fuzzy HyperGraph, Intuitionistic Fuzzy SuperHyperGraph

1 | Introduction

1.1 | From Graphs to Superhypergraphs

Traditional graphs encode binary relationships by treating entities as vertices and drawing edges between every pair of connected vertices [1]. To model interactions involving more than two elements at once, *hypergraphs* generalize this notion by permitting hyperedges to span any nonempty subset of the vertex set, thereby capturing higher-order relationships [2, 3]. Pushing this idea further, a *superhypergraph* arises when one repeatedly takes the powerset of the original vertex set. At each iteration, the “vertices” become the subsets formed



Corresponding Author: gulistanm21@yahoo.com



<https://doi.org/10.61356/j.mawa.2026.10635>



Licensee **Multicriteria Algorithms with Applications**. This article is an open-access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<http://creativecommons.org/licenses/by/4.0>).

in the previous step, producing a nested hierarchy of connections that reflects multi-scale structure in the network [4, 5, 6, 7, 8].

Table 1 summarizes the key distinctions among graphs, hypergraphs, and superhypergraphs. Throughout this paper we take n to be a positive integer unless stated otherwise.

TABLE 1. Comparison of Graphs, Hypergraphs, and Superhypergraphs

Concept	Notation	Edge Type	Extension Strategy
Graph	$G = (V, E)$	$E \subseteq \{\{u, v\} \mid u, v \in V, u \neq v\}$	Edges link exactly two vertices.
Hypergraph	$H = (V, E)$	$E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$	Hyperedges may join any nonempty subset of vertices.
Superhypergraph	$\text{SupHyG}^{(n)} = (V_0, V, E)$	$V, E \subseteq \mathcal{P}^n(V_0)$	Iterated powerset generates nested, multi-level connectivity.

1.2 | Capturing Fuzzy Uncertainty in Networks

Real-world networks often involve incomplete, imprecise, or contradictory information. To model such uncertainty, various “soft” set theories have been incorporated into graph-based frameworks. Fuzzy sets assign membership degrees in $[0, 1]$ to elements [9], while intuitionistic fuzzy sets also include degrees of non-membership [10]. Hyperfuzzy sets introduce multi-dimensional membership structures [11], and neutrosophic sets add an indeterminacy component [12]. Plithogenic sets further generalize by handling contradiction degrees among attributes [13, 14]. When adapted to network models, these theories give rise to fuzzy graphs [9], intuitionistic fuzzy graphs [15], neutrosophic graphs [16], plithogenic graphs [14], and soft graphs [17], each integrating graded or multi-valued relationships to reflect uncertainty more faithfully.

Table 2 provides an overview of fuzzy and intuitionistic fuzzy graph variants. As described below, this paper examines intuitionistic fuzzy superhypergraphs.

TABLE 2. Overview of Fuzzy and Intuitionistic Fuzzy Graph Variants

Graph Type	Membership / Functions	Key Characteristics
Fuzzy Graph	$\sigma: V \rightarrow [0, 1], \mu: V \times V \rightarrow [0, 1]$	Single-valued edge grades, $\mu(u, v) \leq \min\{\sigma(u), \sigma(v)\}$.
Intuitionistic Fuzzy Graph	$\sigma, \sigma^c: V \rightarrow [0, 1], (\mu, \nu): V \times V \rightarrow [0, 1]^2$	Edges have membership μ and non-membership ν , with $\mu + \nu \leq 1$.
Fuzzy HyperGraph	$\sigma: V \rightarrow [0, 1], \mu: E \rightarrow [0, 1]$	Hyperedges connect any subset of vertices, $\mu(e) \leq \min_{v \in e} \sigma(v)$.
Intuitionistic Fuzzy HyperGraph	$\sigma, \sigma^c: V \rightarrow [0, 1], (\mu, \nu): E \rightarrow [0, 1]^2$	Hyperedges with both membership μ and non-membership ν , $\mu + \nu \leq 1$.
Fuzzy SuperHyperGraph	$\sigma: V \rightarrow [0, 1], \mu: E \rightarrow [0, 1]$	n -layer nested hyperedges, graded membership with appurtenance across layers.
Intuitionistic Fuzzy SuperHyperGraph	$\sigma, \sigma^c: V \rightarrow [0, 1], (\mu, \nu): E \rightarrow [0, 1]^2$	Hierarchical superedges with membership μ , non-membership ν , Atanassov and appurtenance constraints.

1.3 | Motivation and Contributions of This Paper

Although uncertain graphs, hypergraphs, and superhypergraphs have been extensively studied for their ability to model complex and imprecise relationships, research on intuitionistic fuzzy superhypergraphs remains scarce and their mathematical properties are not yet well understood. Moreover, many real-world networks exhibit

deeper hierarchical and multi-scale structures that standard hypergraphs cannot fully capture, underscoring the importance of superhypergraph-based approaches.

In this paper, we introduce the concept of an *intuitionistic fuzzy superhypergraph*, which simultaneously incorporates hierarchical layering and intuitionistic fuzziness into a unified framework. We prove that this new structure strictly generalizes both fuzzy superhypergraphs and intuitionistic fuzzy hypergraphs. By doing so, we provide a mathematically rigorous yet intuitive model for representing complex, uncertain networks, and we discuss potential applications in decision-making processes where hierarchical uncertainty plays a critical role. A comparison with existing concepts is provided in Table 2.

Furthermore, we develop and analyze an algorithm for constructing intuitionistic fuzzy superhypergraphs, and we propose a decision-making procedure based on this new model.

1.4 | Structure of this Paper

We organize the remainder of the paper as follows. In Section 2, we review the fundamental definitions of hypergraphs, superhypergraphs, fuzzy superhypergraphs, and intuitionistic fuzzy hypergraphs. Section 3 introduces the formal definition of intuitionistic fuzzy superhypergraphs and illustrates the concept with detailed examples. In Section 4, we present and analyze an algorithm for constructing intuitionistic fuzzy superhypergraphs. Section 5 discusses a decision-making procedure based on intuitionistic fuzzy superhypergraphs. Section 6 offers a concise comparison with existing models and reflects on their relative strengths. Finally, Section 7 concludes the paper and outlines directions for future research.

2 | Preliminaries

This section collects the basic definitions and notation used throughout. Unless otherwise specified, all graphs are assumed to be finite and simple.

2.1 | SuperHyperGraphs

A *hypergraph* extends the notion of a graph by allowing edges (called *hyperedges*) to join any nonempty subset of vertices [18]. Building on this, a *SuperHyperGraph* is obtained by repeatedly applying the powerset operation to the vertex set, producing multiple tiers of connectivity that reveal nested, hierarchical relationships [19, 6].

Definition 1 (Base Set). Let S be a nonempty set. We call S the *base set* from which all subsequent constructions—such as powersets and hyperstructures—are derived.

$$S = \{x \mid x \text{ belongs to the domain of interest}\}.$$

Definition 2 (Powerset). For any set S , its *powerset* $\text{POWS}(S)$ is the collection of all subsets of S :

$$\text{POWS}(S) = \{A \mid A \subseteq S\}.$$

Definition 3 (n th-Iterated Powerset). [20] Given a set H and an integer $n \geq 1$, define

$$\text{POWS}^1(H) = \text{POWS}(H), \quad \text{POWS}^{k+1}(H) = \text{POWS}(\text{POWS}^k(H)).$$

We sometimes write $\text{POWS}_n(H)$ or omit the superscript when the context is clear.

Example 4 (Hierarchical Sensor Network). Consider a set of sensor nodes

$$H = \{S_1, S_2, S_3\}.$$

Then

$$\text{POWS}^1(H) = \{\{S_1, S_2\}, \{S_2, S_3\}, \dots\}$$

represents clusters of sensors within direct communication range, and

$$\text{POWS}^2(H) = \{\{\{S_1, S_2\}, \{S_2, S_3\}\}, \dots\}$$

represents regions composed of these clusters. More generally, $\text{POWS}^n(H)$ models n -level groupings—ranging from individual devices to nested clusters and regions—thus enabling multi-scale network analysis.

Definition 5 (Hypergraph). [3] A *hypergraph* $H = (V, E)$ consists of

- a finite vertex set V ,
- a collection $E \subseteq \text{POWS}(V)$ of nonempty subsets called hyperedges.

Example 6 (Market Basket Hypergraph). Consider an online retailer whose product catalog is

$$V = \{\text{Milk, Bread, Eggs, Butter, ...}\}.$$

Each customer purchase transaction corresponds to a hyperedge. For example,

$$E = \{\{\text{Milk, Bread}\}, \{\text{Bread, Eggs, Butter}\}, \dots\}.$$

In this hypergraph, each hyperedge connects all products bought together in a single transaction, enabling analysis of multi-item purchasing patterns and product affinities.

Definition 7 (SuperHyperGraph). [4] Let V_0 be a nonempty finite set (the *base set*), and let $n \in \mathbb{N}$ with $n \geq 1$. Write $\text{POWS}^k(V_0)$ for the k -fold iterated powerset of V_0 , defined by

$$\text{POWS}^0(V_0) := V_0, \quad \text{POWS}^{k+1}(V_0) := \text{POWS}(\text{POWS}^k(V_0)) \quad (k \geq 0).$$

An n -SuperHyperGraph on V_0 is a triple

$$\text{SupHyG}^{(n)} = (V_0, V, E)$$

consisting of

- (a) a nonempty set of n -supervertices

$$V \subseteq \text{POWS}^n(V_0);$$

- (b) a family of n -superhyperedges

$$E \subseteq \text{POWS}(V) \setminus \{\emptyset\},$$

i.e. each $e \in E$ is a nonempty subset of V .

The elements of V are called n -supervertices, and the elements $e \in E$ are called n -superedges. Thus a SuperHyperGraph is a hypergraph whose vertex set itself consists of iterated-powerset elements of the base set V_0 .

Example 8 (A 3-SuperHyperGraph in collaboration). Let

$$V_0 = \{\text{Ziro, Yuta, Ayano}\}$$

be the finite base set.

First, form some level-1 subsets (teams), which are elements of $\text{POWS}^1(V_0) = \mathcal{P}(V_0)$:

$$T_1 := \{\text{Ziro, Yuta}\}, \quad T_2 := \{\text{Yuta, Ayano}\}.$$

Next, form level-2 subsets (projects) as subsets of $\{T_1, T_2\}$, hence elements of $\text{POWS}^2(V_0) = \mathcal{P}(\text{POWS}^1(V_0))$:

$$P_1 := \{T_1, T_2\}, \quad P_2 := \{T_2\}.$$

Now define level-3 subsets (collaboration portfolios) as subsets of $\{P_1, P_2\}$, so that they lie in

$$\text{POWS}^3(V_0) = \mathcal{P}(\text{POWS}^2(V_0)).$$

For instance, set

$$F_1 := \{P_1\}, \quad F_2 := \{P_1, P_2\}.$$

Then $F_1, F_2 \in \text{POWS}^3(V_0)$.

Define the 3-supervertex set

$$V := \{F_1, F_2\} \subseteq \text{POWS}^3(V_0),$$

and the nonempty 3-superedge

$$E_1 := \{F_1, F_2\} \subseteq V, \quad E := \{E_1\} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

By Definition 7, the triple

$$\text{SupHyG}^{(3)} = (V_0, V, E)$$

is a 3-SuperHyperGraph on the base set V_0 . It models a three-layer collaboration structure: individual members (level 0) form teams (level 1), teams form projects (level 2), and projects are grouped into collaboration portfolios (level 3), which are linked by the superedge E_1 .

A conceptual diagram of a 3-SuperHyperGraph in collaboration is presented in Figure 1.

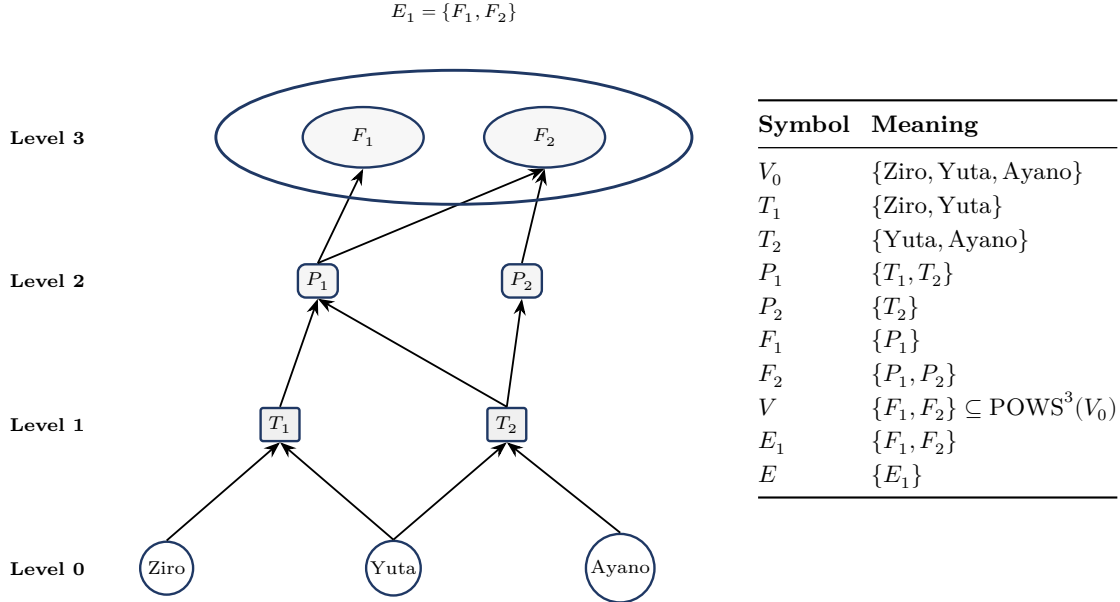


FIGURE 1. A 3-SuperHyperGraph in collaboration (Example 8).

2.2 | Fuzzy n -SuperHyperGraphs

A fuzzy set on a universe U is a function $f: U \rightarrow [0, 1]$, where $f(x)$ represents the degree to which x belongs to the set [21]. Extending this idea, a fuzzy graph endows both vertices and edges with membership values [9], and a fuzzy hypergraph further allows hyperedges to carry such grades [22]. We now adapt these notions to n -SuperHyperGraphs.

Definition 9 (Fuzzy n -SuperHyperGraph). Let

$$\text{SupHyG}^{(n)} = (V_0, V, E)$$

be an n -SuperHyperGraph on the finite base set V_0 in the sense of Definition 7 (so $V \subseteq \text{POWS}^n(V_0)$ and $E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$). A *fuzzy n -SuperHyperGraph* on $\text{SupHyG}^{(n)}$ is a structure

$$(V_0, V, E, \sigma, \mu),$$

often denoted simply by (V, E, σ, μ) when the base set V_0 is understood, where

- $\sigma: V \rightarrow [0, 1]$ assigns to each n -supervertex $v \in V$ a membership degree $\sigma(v)$;
- $\mu: E \rightarrow [0, 1]$ assigns to each n -superedge $e \in E$ a membership degree $\mu(e)$.

These functions must satisfy the *appurtenance condition*

$$\mu(e) \leq \min_{v \in e} \sigma(v), \quad \forall e \in E,$$

i.e. the membership degree of each n -superedge cannot exceed the minimum membership degree of its incident n -supervertices.

Example 10 (Smart home as a fuzzy 2-SuperHyperGraph). Consider three devices

$$V_0 = \{\text{Thermostat}, \text{LightSensor}, \text{SecurityCamera}\}.$$

First-level groupings (rooms) are subsets of V_0 , i.e. elements of $\text{POWS}^1(V_0) = \mathcal{P}(V_0)$:

$$R_1 = \{\text{Thermostat}, \text{LightSensor}\}, \quad R_2 = \{\text{LightSensor}, \text{SecurityCamera}\}.$$

Second-level groupings (zones) are subsets of $\{R_1, R_2\}$, hence elements of the second iterated powerset

$$\text{POWS}^2(V_0) = \mathcal{P}(\text{POWS}^1(V_0)).$$

For example, define

$$Z_1 = \{R_1, R_2\}, \quad Z_2 = \{R_2\},$$

so that $Z_1, Z_2 \in \text{POWS}^2(V_0)$.

Set the 2-supervertex set

$$V = \{Z_1, Z_2\} \subseteq \text{POWS}^2(V_0),$$

and define the single 2-superedge

$$E_1 = \{Z_1, Z_2\} \subseteq V, \quad E = \{E_1\} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

Then

$$\text{SupHyG}^{(2)} = (V_0, V, E)$$

is a 2-SuperHyperGraph on V_0 .

Now assign fuzzy membership values to supervertices and the superedge:

$$\sigma(Z_1) = 0.90, \quad \sigma(Z_2) = 0.80, \quad \mu(E_1) = 0.75.$$

We check the appurtenance condition:

$$0.75 = \mu(E_1) \leq \min\{\sigma(Z_1), \sigma(Z_2)\} = \min\{0.90, 0.80\} = 0.80.$$

Thus the inequality holds, and

$$(V_0, V, E, \sigma, \mu)$$

is a valid fuzzy 2-SuperHyperGraph modeling hierarchical device reliability in the smart home.

2.3 | Intuitionistic Fuzzy Hypergraph

An *intuitionistic fuzzy set* assigns to each element both a membership degree and a nonmembership degree, satisfying Atanassov's constraint $0 \leq \mu(x) + \nu(x) \leq 1$ [12]. An *intuitionistic fuzzy graph* extends a classical graph by assigning every vertex and every edge both membership and nonmembership values, also respecting Atanassov's sum constraint [23, 24]. An *intuitionistic fuzzy hypergraph* generalizes hypergraphs by equipping each vertex and each hyperedge with membership and nonmembership degrees under the same Atanassov framework [25].

Definition 11 (Intuitionistic Fuzzy Hypergraph). (cf. [25, 26]) Let V be a nonempty finite set of *vertices*. An *intuitionistic fuzzy hyperedge* on V is an ordered pair

$$E = (\mu_E, \nu_E),$$

where

$$\mu_E, \nu_E: V \longrightarrow [0, 1]$$

such that

$$0 \leq \mu_E(v) + \nu_E(v) \leq 1, \quad \forall v \in V.$$

Its *support* is defined by

$$\text{supp}(E) = \{v \in V \mid \mu_E(v) > 0 \text{ or } \nu_E(v) < 1\}.$$

An *intuitionistic fuzzy hypergraph* is a pair

$$H = (V, \mathcal{E}),$$

where $\mathcal{E} = \{E_1, \dots, E_m\}$ is a finite family of intuitionistic fuzzy hyperedges on V satisfying the covering condition

$$\bigcup_{j=1}^m \text{supp}(E_j) = V.$$

The elements of V are called *vertices*, and each $E_j \in \mathcal{E}$ is called an *intuitionistic fuzzy hyperedge*. The *order* of H is $|V|$, and the number of hyperedges is $|\mathcal{E}|$.

Example 12 (Medical Diagnosis as an Intuitionistic Fuzzy Hypergraph). Let

$$V = \{\text{Fever, Cough, Fatigue, Headache}\}.$$

We model two diseases—“Flu” and “COVID19”—as intuitionistic fuzzy hyperedges E_1 and E_2 via functions $\mu_{E_i}, \nu_{E_i} : V \rightarrow [0, 1]$ defined by

$$\begin{aligned} \mu_{E_1}(\text{Fever}) &= 0.90, & \nu_{E_1}(\text{Fever}) &= 0.05, \\ \mu_{E_1}(\text{Cough}) &= 0.80, & \nu_{E_1}(\text{Cough}) &= 0.10, \\ \mu_{E_1}(\text{Fatigue}) &= 0.75, & \nu_{E_1}(\text{Fatigue}) &= 0.15, \\ \mu_{E_1}(\text{Headache}) &= 0, & \nu_{E_1}(\text{Headache}) &= 1.00; \\ \mu_{E_2}(\text{Fever}) &= 0, & \nu_{E_2}(\text{Fever}) &= 1.00, \\ \mu_{E_2}(\text{Cough}) &= 0.85, & \nu_{E_2}(\text{Cough}) &= 0.05, \\ \mu_{E_2}(\text{Fatigue}) &= 0.70, & \nu_{E_2}(\text{Fatigue}) &= 0.20, \\ \mu_{E_2}(\text{Headache}) &= 0.80, & \nu_{E_2}(\text{Headache}) &= 0.10. \end{aligned}$$

For each $i = 1, 2$ and $v \in V$ we have

$$0 \leq \mu_{E_i}(v) + \nu_{E_i}(v) \leq 1,$$

so both E_1 and E_2 are intuitionistic fuzzy hyperedges on V . Their supports are

$$\text{supp}(E_1) = \{\text{Fever, Cough, Fatigue}\}, \quad \text{supp}(E_2) = \{\text{Cough, Fatigue, Headache}\},$$

and thus

$$\text{supp}(E_1) \cup \text{supp}(E_2) = V.$$

Therefore

$$H = (V, \{E_1, E_2\})$$

is an intuitionistic fuzzy hypergraph that captures both diagnostic confidence (membership) and hesitation (nonmembership) for each symptom under the two diseases.

3 | Main Results: Intuitionistic Superhypergraphs

An intuitionistic superhypergraph is a hierarchical hypergraph in which every n -supervertex and superedge is equipped with both membership and non-membership degrees that satisfy Atanassov’s sum and appurtenance constraints. We now give the formal definition of this structure.

Definition 13 (Intuitionistic fuzzy n -SuperHyperGraph). Let

$$\text{SupHyG}^{(n)} = (V_0, V, E)$$

be an n -SuperHyperGraph on the finite base set V_0 in the sense of Definition 7, so that $V \subseteq \text{POWS}^n(V_0)$ and $E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$. An *intuitionistic fuzzy n -SuperHyperGraph* on $\text{SupHyG}^{(n)}$ is a sextuple

$$\mathcal{H} = (V, E, \sigma, \sigma^c, \mu, \nu),$$

where the functions

$$\begin{aligned} \sigma : V &\longrightarrow [0, 1], & \sigma^c : V &\longrightarrow [0, 1], \\ \mu : E &\longrightarrow [0, 1], & \nu : E &\longrightarrow [0, 1], \end{aligned}$$

satisfy, for all $v \in V$ and $e \in E$,

$$0 \leq \sigma(v) + \sigma^c(v) \leq 1, \quad 0 \leq \mu(e) + \nu(e) \leq 1, \quad (1)$$

$$\mu(e) \leq \min_{v \in e} \sigma(v), \quad \nu(e) \leq \min_{v \in e} \sigma^c(v). \quad (2)$$

Here $\sigma(v)$ and $\sigma^c(v)$ are the membership and non-membership degrees of the n -supervertex v , while $\mu(e)$ and $\nu(e)$ are the membership and non-membership degrees of the n -superedge e . Equation (1) is the Atanassov constraint, and (2) enforces consistency of edge-values with those of their incident supervertices.

Example 14 (Multi-level team-department collaboration). Consider a company with three employees

$$V_0 = \{\text{Ziro, Yuta, Ayano}\}.$$

First-level *teams* are subsets of V_0 , i.e. elements of $\text{POWS}^1(V_0) = \mathcal{P}(V_0)$. For instance, let

$$R_1 = \{\text{Ziro, Yuta}\}, \quad R_2 = \{\text{Yuta, Ayano}\}.$$

Second-level *departments* are subsets of $\{R_1, R_2\}$, hence elements of the second iterated powerset

$$\text{POWS}^2(V_0) = \mathcal{P}(\text{POWS}^1(V_0)).$$

Define

$$D_1 = \{R_1, R_2\}, \quad D_2 = \{R_2\},$$

so $D_1, D_2 \in \text{POWS}^2(V_0)$, and set the 2-supervertex set

$$V = \{D_1, D_2\} \subseteq \text{POWS}^2(V_0).$$

Introduce a single 2-superedge

$$E_1 = \{D_1, D_2\} \subseteq V, \quad E = \{E_1\} \subseteq \mathcal{P}(V) \setminus \{\emptyset\},$$

so that

$$\text{SupHyG}^{(2)} = (V_0, V, E)$$

is a 2-SuperHyperGraph.

We now endow it with intuitionistic fuzzy values:

- Membership and non-membership of departments (2-supervertices):

$$\sigma(D_1) = 0.90, \quad \sigma^c(D_1) = 0.05; \quad \sigma(D_2) = 0.85, \quad \sigma^c(D_2) = 0.10.$$

Then

$$0 \leq \sigma(D_i) + \sigma^c(D_i) \leq 1 \quad \text{for } i = 1, 2,$$

since $0.90 + 0.05 = 0.95$ and $0.85 + 0.10 = 0.95$.

- Membership and non-membership of the cross-department project (2-superedge):

$$\mu(E_1) = 0.80, \quad \nu(E_1) = 0.05,$$

so

$$0 \leq \mu(E_1) + \nu(E_1) = 0.85 \leq 1.$$

We now check the appurtenance constraints (2):

$$\mu(E_1) = 0.80 \leq \min\{\sigma(D_1), \sigma(D_2)\} = \min\{0.90, 0.85\} = 0.85,$$

$$\nu(E_1) = 0.05 \leq \min\{\sigma^c(D_1), \sigma^c(D_2)\} = \min\{0.05, 0.10\} = 0.05.$$

Thus

$$\mathcal{H} = (V, \{E_1\}, \sigma, \sigma^c, \mu, \nu)$$

is a valid intuitionistic fuzzy 2-SuperHyperGraph modeling the degrees of participation and hesitation of each department in the joint project.

Example 15 (University curriculum as an intuitionistic fuzzy 2-SuperHyperGraph). Consider a small university department offering three base courses

$$V_0 = \{\text{Calculus}, \text{DataStruct}, \text{Algorithms}\}.$$

First-level *modules* are subsets of V_0 , i.e. elements of $\text{POWS}^1(V_0) = \mathcal{P}(V_0)$. For instance, define

$$M_1 = \{\text{Calculus}, \text{DataStruct}\}, \quad M_2 = \{\text{DataStruct}, \text{Algorithms}\}.$$

Second-level *degree programs* are subsets of $\{M_1, M_2\}$, hence elements of the second iterated powerset $\text{POWS}^2(V_0)$. Let

$$P_A = \{M_1, M_2\}, \quad P_B = \{M_2\},$$

and set the 2-supervertex set

$$V = \{P_A, P_B\} \subseteq \text{POWS}^2(V_0).$$

Assume an accreditation review committee evaluates both programs simultaneously; we represent this by the single 2-superedge

$$E_1 = \{P_A, P_B\} \subseteq V, \quad E = \{E_1\} \subseteq \mathcal{P}(V) \setminus \{\emptyset\},$$

so that

$$\text{SupHyG}^{(2)} = (V_0, V, E)$$

is a 2-SuperHyperGraph.

We now assign intuitionistic fuzzy values reflecting program quality (σ, μ) and review uncertainty (σ^c, ν) at the program (2-supervertex) level:

- For degree programs (2-supervertices):

$$\sigma(P_A) = 0.85, \quad \sigma^c(P_A) = 0.05; \quad \sigma(P_B) = 0.80, \quad \sigma^c(P_B) = 0.10.$$

Hence

$$0 \leq \sigma(P_A) + \sigma^c(P_A) = 0.90 \leq 1, \quad 0 \leq \sigma(P_B) + \sigma^c(P_B) = 0.90 \leq 1.$$

- For the joint accreditation review (2-superedge):

$$\mu(E_1) = 0.75, \quad \nu(E_1) = 0.05,$$

so

$$0 \leq \mu(E_1) + \nu(E_1) = 0.80 \leq 1.$$

We verify the appurtenance constraints:

$$\mu(E_1) = 0.75 \leq \min\{\sigma(P_A), \sigma(P_B)\} = \min\{0.85, 0.80\} = 0.80,$$

$$\nu(E_1) = 0.05 \leq \min\{\sigma^c(P_A), \sigma^c(P_B)\} = \min\{0.05, 0.10\} = 0.05.$$

Thus

$$\mathcal{H} = (V, \{E_1\}, \sigma, \sigma^c, \mu, \nu)$$

is a valid intuitionistic fuzzy 2-SuperHyperGraph modeling the curriculum structure and its accreditation uncertainty at the program level.

Example 16 (Multi-level corporate structure as an intuitionistic fuzzy 3-SuperHyperGraph). Consider a company with four employees

$$V_0 = \{\text{Hayato}, \text{Ayana}, \text{Yuya}, \text{Dave}\}.$$

We form three hierarchical grouping levels:

- *Teams* (level 1) are subsets of V_0 , i.e. elements of $\text{POWS}^1(V_0)$:

$$T_1 = \{\text{Hayato}, \text{Ayana}\}, \quad T_2 = \{\text{Yuya}, \text{Dave}\}.$$

- *Departments* (level 2) are subsets of $\{T_1, T_2\}$, hence elements of $\text{POWS}^2(V_0)$:

$$D_1 = \{T_1\}, \quad D_2 = \{T_2\}, \quad D_3 = \{T_1, T_2\}.$$

- *Divisions* (level 3) are subsets of $\{D_1, D_2, D_3\}$, hence elements of $\text{POWS}^3(V_0)$. Define

$$\text{Div}_1 = \{D_1, D_3\}, \quad \text{Div}_2 = \{D_2, D_3\},$$

and set

$$V = \{\text{Div}_1, \text{Div}_2\} \subseteq \text{POWS}^3(V_0).$$

Suppose there is a company-wide initiative spanning both divisions, represented by the single 3-superedge

$$E_1 = \{\text{Div}_1, \text{Div}_2\} \subseteq V, \quad E = \{E_1\} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

Then

$$\text{SupHyG}^{(3)} = (V_0, V, E)$$

is a 3-SuperHyperGraph on V_0 .

We assign intuitionistic fuzzy values indicating each division's engagement (σ, σ^c) and the initiative's overall coherence (μ, ν) :

- **Divisional membership/non-membership (3-supervertices):**

$$\sigma(\text{Div}_1) = 0.95, \quad \sigma^c(\text{Div}_1) = 0.02; \quad \sigma(\text{Div}_2) = 0.90, \quad \sigma^c(\text{Div}_2) = 0.05.$$

Hence

$$0 \leq \sigma(\text{Div}_1) + \sigma^c(\text{Div}_1) = 0.97 \leq 1, \quad 0 \leq \sigma(\text{Div}_2) + \sigma^c(\text{Div}_2) = 0.95 \leq 1.$$

- **Initiative membership/non-membership (3-superedge):**

$$\mu(E_1) = 0.85, \quad \nu(E_1) = 0.02,$$

so

$$0 \leq \mu(E_1) + \nu(E_1) = 0.87 \leq 1.$$

Finally, we verify the appurtenance constraints (2):

$$\mu(E_1) = 0.85 \leq \min\{\sigma(\text{Div}_1), \sigma(\text{Div}_2)\} = \min\{0.95, 0.90\} = 0.90,$$

$$\nu(E_1) = 0.02 \leq \min\{\sigma^c(\text{Div}_1), \sigma^c(\text{Div}_2)\} = \min\{0.02, 0.05\} = 0.02.$$

Therefore

$$\mathcal{H} = (V, \{E_1\}, \sigma, \sigma^c, \mu, \nu)$$

is a valid intuitionistic fuzzy 3-SuperHyperGraph modeling the degrees of engagement and hesitation of each division in the cross-division initiative.

Theorem 17. *The class of intuitionistic fuzzy n -superhypergraphs strictly generalizes both*

- (1) *the class of fuzzy n -superhypergraphs (by “switching off” non-membership), and*
- (2) *the class of intuitionistic fuzzy hypergraphs (by choosing $n = 1$ and restricting to singleton supervertices).*

Proof: (1) **Recovering fuzzy n -superhypergraphs.** Let (V, E, σ, μ) be a fuzzy n -superhypergraph, so that

$$\sigma: V \longrightarrow [0, 1], \quad \mu: E \longrightarrow [0, 1],$$

and for all $e \in E$,

$$\mu(e) \leq \min_{v \in e} \sigma(v).$$

Define

$$\sigma^c(v) := 0 \quad (v \in V), \quad \nu(e) := 0 \quad (e \in E).$$

Then for every $v \in V$ and $e \in E$,

$$0 \leq \sigma(v) + \sigma^c(v) = \sigma(v) \leq 1, \quad 0 \leq \mu(e) + \nu(e) = \mu(e) \leq 1,$$

and the appurtenance constraints become

$$\mu(e) \leq \min_{v \in e} \sigma(v), \quad \nu(e) = 0 \leq \min_{v \in e} \sigma^c(v) = 0.$$

Hence $\mathcal{H} = (V, E, \sigma, \sigma^c, \mu, \nu)$ is an intuitionistic fuzzy n -superhypergraph whose underlying fuzzy structure is exactly (V, E, σ, μ) .

Conversely, if $\mathcal{H} = (V, E, \sigma, \sigma^c, \mu, \nu)$ is an intuitionistic fuzzy n -superhypergraph with $\sigma^c \equiv 0$ and $\nu \equiv 0$, then the Atanassov constraint (1) reduces to $0 \leq \sigma(v) \leq 1$ and $0 \leq \mu(e) \leq 1$, and (2) reduces to $\mu(e) \leq \min_{v \in e} \sigma(v)$, which is precisely the appurtenance condition of a fuzzy n -superhypergraph (V, E, σ, μ) . Thus fuzzy n -superhypergraphs form a proper subclass of intuitionistic fuzzy n -superhypergraphs.

(2) Recovering intuitionistic fuzzy hypergraphs. Let

$$(V_0, \mathcal{E}, \mu_H, \nu_H, \mu_E, \nu_E)$$

be an intuitionistic fuzzy hypergraph as in Definition 11, so $\mathcal{E} \subseteq \mathcal{P}(V_0)$, and for all $v \in V_0$, $e \in \mathcal{E}$,

$$0 \leq \mu_H(v) + \nu_H(v) \leq 1, \quad 0 \leq \mu_E(e) + \nu_E(e) \leq 1,$$

with appurtenance constraints

$$\mu_E(e) \leq \min_{v \in e} \mu_H(v), \quad \nu_E(e) \leq \min_{v \in e} \nu_H(v).$$

Set $n = 1$. Define the level-1 supervertex set

$$V := \{\{v\} \mid v \in V_0\} \subseteq \text{POWS}^1(V_0),$$

and for each hyperedge $e \in \mathcal{E}$ define the corresponding 1-superedge

$$\hat{e} := \{\{v\} \mid v \in e\} \subseteq V.$$

Let

$$E := \{\hat{e} \mid e \in \mathcal{E}\}.$$

Then (V, E) is a 1-superhypergraph on the base set V_0 .

Define functions on V and E by

$$\begin{aligned} \sigma(\{v\}) &:= \mu_H(v), & \sigma^c(\{v\}) &:= \nu_H(v) & (v \in V_0), \\ \mu(\hat{e}) &:= \mu_E(e), & \nu(\hat{e}) &:= \nu_E(e) & (e \in \mathcal{E}). \end{aligned}$$

For any $v \in V_0$,

$$0 \leq \sigma(\{v\}) + \sigma^c(\{v\}) = \mu_H(v) + \nu_H(v) \leq 1,$$

and for any $e \in \mathcal{E}$,

$$0 \leq \mu(\hat{e}) + \nu(\hat{e}) = \mu_E(e) + \nu_E(e) \leq 1.$$

For appurtenance, if $\hat{e} \in E$, then

$$\begin{aligned} \mu(\hat{e}) &= \mu_E(e) \leq \min_{v \in e} \mu_H(v) = \min_{u \in \hat{e}} \sigma(u), \\ \nu(\hat{e}) &= \nu_E(e) \leq \min_{v \in e} \nu_H(v) = \min_{u \in \hat{e}} \sigma^c(u), \end{aligned}$$

so $(V, E, \sigma, \sigma^c, \mu, \nu)$ is an intuitionistic fuzzy 1-superhypergraph whose vertices are exactly the singleton supervertices $\{v\}$ with $v \in V_0$.

Conversely, any intuitionistic fuzzy 1-superhypergraph whose supervertex set consists only of singletons of V_0 and whose superedges are collections of these singletons corresponds uniquely to an intuitionistic fuzzy hypergraph on V_0 by the inverse construction

$$e := \{v \in V_0 \mid \{v\} \in \hat{e}\} \quad \text{for each } \hat{e} \in E.$$

Finally, intuitionistic fuzzy n -superhypergraphs with nontrivial non-membership functions (that is, some $\sigma^c \not\equiv 0$ or $\nu \not\equiv 0$) or with $n > 1$ and genuinely higher-level supervertices (such as those in Examples 14 and 16) are not fuzzy n -superhypergraphs and are not intuitionistic fuzzy hypergraphs. Hence the inclusions in (1) and (2) are proper, so the generalization is strict. $\square$

Theorem 18 (Level-Cut n -Superhypergraph). *Let $\mathcal{H} = (V, E, \sigma, \sigma^c, \mu, \nu)$ be an intuitionistic fuzzy n -superhypergraph. For any $\alpha \in [0, 1]$, define the α -cut of $\mathcal{H}$ as the crisp pair*

$$V_\alpha := \{v \in V \mid \sigma(v) \geq \alpha\}, \quad E_\alpha := \{e \in E \mid \mu(e) \geq \alpha\}.$$

Then (V_α, E_α) is an n -superhypergraph (with the same underlying base set as $\mathcal{H}$).

Proof: Because (V, E) is an n -superhypergraph, every $v \in V$ lies in $\text{POWS}^n(V_0)$ and every $e \in E$ is a nonempty subset of V . Hence $V_\alpha \subseteq V \subseteq \text{POWS}^n(V_0)$ and $E_\alpha \subseteq E$, so all elements of E_α are nonempty subsets of V .

Let $e \in E_\alpha$. Then $\mu(e) \geq \alpha$. From the appurtenance constraint (2) we have

$$\mu(e) \leq \min_{v \in e} \sigma(v).$$

Thus

$$\min_{v \in e} \sigma(v) \geq \mu(e) \geq \alpha,$$

which implies $\sigma(v) \geq \alpha$ for every $v \in e$. Therefore $e \subseteq V_\alpha$, and hence $E_\alpha \subseteq \mathcal{P}(V_\alpha) \setminus \{\emptyset\}$. All structural requirements of the underlying crisp n -superhypergraph are therefore inherited, so (V_α, E_α) is an n -superhypergraph on the restricted vertex set V_α . $\square$

Theorem 19 (Closure under Intersection). *Let $\mathcal{H}_i = (V, E, \sigma_i, \sigma_i^c, \mu_i, \nu_i)$ for $i = 1, 2$ be two intuitionistic fuzzy n -superhypergraphs on the same underlying n -superhypergraph (V, E) . Define*

$$\begin{aligned} \sigma(v) &:= \min\{\sigma_1(v), \sigma_2(v)\}, & \sigma^c(v) &:= \max\{\sigma_1^c(v), \sigma_2^c(v)\}, \\ \mu(e) &:= \min\{\mu_1(e), \mu_2(e)\}, & \nu(e) &:= \max\{\nu_1(e), \nu_2(e)\}. \end{aligned}$$

Then $\mathcal{H} = (V, E, \sigma, \sigma^c, \mu, \nu)$ is also an intuitionistic fuzzy n -superhypergraph.

Proof: Fix $v \in V$. Set

$$a_1 := \sigma_1(v), \quad a_2 := \sigma_2(v), \quad b_1 := \sigma_1^c(v), \quad b_2 := \sigma_2^c(v).$$

Then

$$\sigma(v) = \min(a_1, a_2), \quad \sigma^c(v) = \max(b_1, b_2).$$

By the Atanassov constraint for $\mathcal{H}_1$ and $\mathcal{H}_2$,

$$0 \leq a_i + b_i \leq 1 \quad (i = 1, 2).$$

For any real numbers x_1, x_2, y_1, y_2 ,

$$\min(x_1, x_2) + \max(y_1, y_2) \leq \max(x_1 + y_1, x_2 + y_2),$$

because if $x_1 \leq x_2$ and $y_1 \geq y_2$, the left side equals $x_1 + y_1$, and if $x_1 \leq x_2$ and $y_2 > y_1$, then $\min(x_1, x_2) + \max(y_1, y_2) = x_1 + y_2 \leq x_2 + y_2$. Applying this with $x_i = a_i$, $y_i = b_i$ gives

$$\sigma(v) + \sigma^c(v) = \min(a_1, a_2) + \max(b_1, b_2) \leq \max(a_1 + b_1, a_2 + b_2) \leq 1.$$

Thus $0 \leq \sigma(v) + \sigma^c(v) \leq 1$ for all $v \in V$.

An identical argument with $\mu_i(e)$ and $\nu_i(e)$ in place of $\sigma_i(v)$ and $\sigma_i^c(v)$ shows that $0 \leq \mu(e) + \nu(e) \leq 1$ for all $e \in E$.

Now fix $e \in E$. From the appurtenance constraints in $\mathcal{H}_1$ and $\mathcal{H}_2$,

$$\mu_i(e) \leq \min_{v \in e} \sigma_i(v) \quad (i = 1, 2).$$

Hence

$$\mu(e) = \min(\mu_1(e), \mu_2(e)) \leq \min\left(\min_{v \in e} \sigma_1(v), \min_{v \in e} \sigma_2(v)\right).$$

For any functions f, g on a finite set,

$$\min_x (\min_x f(x), \min_x g(x)) = \min_x \min(f(x), g(x)).$$

Applying this with $f(v) = \sigma_1(v)$, $g(v) = \sigma_2(v)$, we obtain

$$\mu(e) \leq \min_{v \in e} \min(\sigma_1(v), \sigma_2(v)) = \min_{v \in e} \sigma(v).$$

Similarly, from $\nu_i(e) \leq \min_{v \in e} \sigma_i^c(v)$ we get

$$\nu(e) = \max(\nu_1(e), \nu_2(e)) \leq \max\left(\min_{v \in e} \sigma_1^c(v), \min_{v \in e} \sigma_2^c(v)\right).$$

For any nonnegative functions f, g ,

$$\max\left(\min_x f(x), \min_x g(x)\right) \leq \min_x \max(f(x), g(x)),$$

because $\max(f(x), g(x)) \geq f(x)$ and $\geq g(x)$ pointwise, so its minimum is at least both $\min_x f(x)$ and $\min_x g(x)$. Taking $f(v) = \sigma_1^c(v)$, $g(v) = \sigma_2^c(v)$ gives

$$\nu(e) \leq \min_{v \in e} \max(\sigma_1^c(v), \sigma_2^c(v)) = \min_{v \in e} \sigma^c(v).$$

Thus both appurtenance inequalities in (2) hold for $\mathcal{H}$, so $\mathcal{H}$ is an intuitionistic fuzzy n -superhypergraph. $\square$

Theorem 20 (Complementarity). *Let $\mathcal{H} = (V, E, \sigma, \sigma^c, \mu, \nu)$ be an intuitionistic fuzzy n -superhypergraph. Define its complement by*

$$\sigma'(v) := \sigma^c(v), \quad \sigma'^c(v) := \sigma(v), \quad \mu'(e) := \nu(e), \quad \nu'(e) := \mu(e).$$

Then $\mathcal{H}' = (V, E, \sigma', \sigma'^c, \mu', \nu')$ is also an intuitionistic fuzzy n -superhypergraph.

Proof: For all $v \in V$,

$$0 \leq \sigma(v) + \sigma^c(v) \leq 1 \implies 0 \leq \sigma'(v) + \sigma'^c(v) = \sigma^c(v) + \sigma(v) \leq 1,$$

so the Atanassov constraint (1) holds for vertices of $\mathcal{H}'$. Similarly, for all $e \in E$,

$$0 \leq \mu(e) + \nu(e) \leq 1 \implies 0 \leq \mu'(e) + \nu'(e) = \nu(e) + \mu(e) \leq 1.$$

For appurtenance, if $e \in E$,

$$\mu'(e) = \nu(e) \leq \min_{v \in e} \sigma^c(v) = \min_{v \in e} \sigma'(v),$$

$$\nu'(e) = \mu(e) \leq \min_{v \in e} \sigma(v) = \min_{v \in e} \sigma'^c(v),$$

again by (2). Hence $\mathcal{H}'$ satisfies all axioms of an intuitionistic fuzzy n -superhypergraph. $\square$

Theorem 21 (Closure under Union). *Let $\mathcal{H}_i = (V, E, \sigma_i, \sigma_i^c, \mu_i, \nu_i)$ for $i = 1, 2$ be two intuitionistic fuzzy n -superhypergraphs on the same underlying n -superhypergraph (V, E) . Define their union by*

$$\sigma(v) := \max\{\sigma_1(v), \sigma_2(v)\}, \quad \sigma^c(v) := \min\{\sigma_1^c(v), \sigma_2^c(v)\},$$

$$\mu(e) := \max\{\mu_1(e), \mu_2(e)\}, \quad \nu(e) := \min\{\nu_1(e), \nu_2(e)\}.$$

Then $\mathcal{H} = (V, E, \sigma, \sigma^c, \mu, \nu)$ is an intuitionistic fuzzy n -superhypergraph.

Proof: Fix $v \in V$ and set

$$a_1 := \sigma_1(v), \quad a_2 := \sigma_2(v), \quad b_1 := \sigma_1^c(v), \quad b_2 := \sigma_2^c(v).$$

Then

$$\sigma(v) = \max(a_1, a_2), \quad \sigma^c(v) = \min(b_1, b_2).$$

By the Atanassov constraint for $\mathcal{H}_1$ and $\mathcal{H}_2$,

$$0 \leq a_i + b_i \leq 1 \quad (i = 1, 2).$$

For any real numbers x_1, x_2, y_1, y_2 ,

$$\max(x_1, x_2) + \min(y_1, y_2) \leq \max(x_1 + y_1, x_2 + y_2),$$

since if $x_1 \geq x_2$ and $y_1 \leq y_2$, the left side equals $x_1 + y_1$, and if $x_1 \geq x_2$ and $y_2 < y_1$, then $\max(x_1, x_2) + \min(y_1, y_2) = x_1 + y_2 \leq x_1 + y_1$. Applying this with $x_i = a_i$, $y_i = b_i$ yields

$$\sigma(v) + \sigma^c(v) = \max(a_1, a_2) + \min(b_1, b_2) \leq \max(a_1 + b_1, a_2 + b_2) \leq 1.$$

Thus $0 \leq \sigma(v) + \sigma^c(v) \leq 1$ for all $v \in V$.

For edges, the same inequality with $\mu_i(e)$ and $\nu_i(e)$ in place of $\sigma_i(v)$ and $\sigma_i^c(v)$ shows that $0 \leq \mu(e) + \nu(e) \leq 1$ for all $e \in E$.

Now fix $e \in E$. From $\mu_i(e) \leq \min_{v \in e} \sigma_i(v)$ we obtain

$$\mu(e) = \max(\mu_1(e), \mu_2(e)) \leq \max\left(\min_{v \in e} \sigma_1(v), \min_{v \in e} \sigma_2(v)\right).$$

For any functions f, g on a finite set,

$$\max\left(\min_x f(x), \min_x g(x)\right) \leq \min_x \max(f(x), g(x)),$$

because $\max(f(x), g(x)) \geq f(x)$ and $\geq g(x)$ pointwise, so its minimum is at least both $\min_x f(x)$ and $\min_x g(x)$. Taking $f(v) = \sigma_1(v)$, $g(v) = \sigma_2(v)$, we get

$$\mu(e) \leq \min_{v \in e} \max(\sigma_1(v), \sigma_2(v)) = \min_{v \in e} \sigma(v).$$

For ν , from $\nu_i(e) \leq \min_{v \in e} \sigma_i^c(v)$ we deduce

$$\nu(e) = \min(\nu_1(e), \nu_2(e)) \leq \min\left(\min_{v \in e} \sigma_1^c(v), \min_{v \in e} \sigma_2^c(v)\right).$$

For any functions f, g ,

$$\min\left(\min_x f(x), \min_x g(x)\right) = \min_x \min(f(x), g(x)),$$

so with $f(v) = \sigma_1^c(v)$, $g(v) = \sigma_2^c(v)$ we have

$$\nu(e) \leq \min_{v \in e} \min(\sigma_1^c(v), \sigma_2^c(v)) = \min_{v \in e} \sigma^c(v).$$

Thus both appurtenance inequalities (2) hold for $\mathcal{H}$, so $\mathcal{H}$ is an intuitionistic fuzzy n -superhypergraph. $\square$

Theorem 22 (Complement Involution). *For any intuitionistic fuzzy n -superhypergraph $\mathcal{H} = (V, E, \sigma, \sigma^c, \mu, \nu)$, let $\mathcal{H}' = (V, E, \sigma', \sigma'^c, \mu', \nu')$ be its complement as in Theorem 20. Then the complement of $\mathcal{H}'$ is $\mathcal{H}$ itself.*

Proof: By Definition 20 of the complement,

$$\sigma'(v) = \sigma^c(v), \quad \sigma'^c(v) = \sigma(v), \quad \mu'(e) = \nu(e), \quad \nu'(e) = \mu(e).$$

Applying the same construction once more gives

$$\begin{aligned} \sigma''(v) &= \sigma'^c(v) = \sigma(v), & \sigma''^c(v) &= \sigma'(v) = \sigma^c(v), \\ \mu''(e) &= \nu'(e) = \mu(e), & \nu''(e) &= \mu'(e) = \nu(e), \end{aligned}$$

for all $v \in V$, $e \in E$. Hence the double complement $\mathcal{H}''$ coincides with $\mathcal{H}$, proving the involution property. $\square$

Theorem 23 (Monotonicity of α -Cuts). *Let $\mathcal{H} = (V, E, \sigma, \sigma^c, \mu, \nu)$ be an intuitionistic fuzzy n -superhypergraph, and for $\alpha \in [0, 1]$ define the α -cut (V_α, E_α) as in Theorem 18. Then for any $0 \leq \alpha \leq \beta \leq 1$,*

$$V_\beta \subseteq V_\alpha, \quad E_\beta \subseteq E_\alpha.$$

Proof: If $v \in V_\beta$, then by definition $\sigma(v) \geq \beta$. Since $\beta \geq \alpha$, we have $\sigma(v) \geq \alpha$, so $v \in V_\alpha$. Hence $V_\beta \subseteq V_\alpha$.

Similarly, if $e \in E_\beta$, then $\mu(e) \geq \beta \geq \alpha$, so $e \in E_\alpha$. Therefore $E_\beta \subseteq E_\alpha$. $\square$

4 | A Simple Graph Algorithm for Constructing Intuitionistic Fuzzy n -Superhypergraphs

We now present a procedure whose running time is linear in the size of the incidence representation of a crisp n -superhypergraph $\text{SuHyG}^{(n)} = (V, E)$. Starting from arbitrary intuitionistic-fuzzy data $\sigma^r, \sigma^{r,c} : V \rightarrow [0, 1]$ and $\mu^r, \nu^r : E \rightarrow [0, 1]$, it constructs a valid intuitionistic fuzzy n -superhypergraph $\mathcal{H} = (V, E, \sigma, \sigma^c, \mu, \nu)$ by enforcing:

(1) **Atanassov's sum constraints:**

$$0 \leq \sigma(v) + \sigma^c(v) \leq 1, \quad 0 \leq \mu(e) + \nu(e) \leq 1, \quad \forall v \in V, e \in E.$$

(2) **Appurtenance constraints:**

$$\mu(e) \leq \min_{v \in e} \sigma(v), \quad \nu(e) \leq \min_{v \in e} \sigma^c(v), \quad \forall e \in E.$$

Algorithm 1: Construct Intuitionistic Fuzzy n -Superhypergraph

Input: crisp n -superhypergraph $\text{SuHyG}^{(n)} = (V, E)$,
raw vertex values $\sigma^r, \sigma^{r,c} : V \rightarrow [0, 1]$,
raw edge values $\mu^r, \nu^r : E \rightarrow [0, 1]$.
Output: intuitionistic fuzzy n -superhypergraph $\mathcal{H} = (V, E, \sigma, \sigma^c, \mu, \nu)$.

```

1  $\sigma \leftarrow \sigma^r, \sigma^c \leftarrow \sigma^{r,c};$ 
2  $\mu \leftarrow \mu^r, \nu \leftarrow \nu^r;$ 
3 foreach  $v \in V$  do
4    $s \leftarrow \sigma(v) + \sigma^c(v);$ 
5   if  $s > 1$  then
6      $\sigma(v) \leftarrow \sigma(v)/s;$ 
7      $\sigma^c(v) \leftarrow \sigma^c(v)/s;$ 
8 foreach  $e \in E$  do
9    $\mu(e) \leftarrow \min\{\mu(e), \min_{v \in e} \sigma(v)\};$ 
10   $\nu(e) \leftarrow \min\{\nu(e), \min_{v \in e} \sigma^c(v)\};$ 
11   $t \leftarrow \mu(e) + \nu(e);$ 
12  if  $t > 1$  then
13     $\mu(e) \leftarrow \mu(e)/t;$ 
14     $\nu(e) \leftarrow \nu(e)/t;$ 
15 return  $(V, E, \sigma, \sigma^c, \mu, \nu);$ 

```

Example 24 (Application to a 2-Superhypergraph). Let $V_0 = \{a, b\}$. The second iterated powerset $\mathcal{P}^2(V_0) = \mathcal{P}(\mathcal{P}(V_0))$ contains, among many other elements, the sets $\{a\}$, $\{b\}$, and $\{a, b\}$. For this example we choose the subset

$$V = \{v_1 = \{a\}, v_2 = \{b\}, v_3 = \{a, b\}\} \subseteq \mathcal{P}^2(V_0), \quad E = \{e_1 = \{v_1, v_2\}\}.$$

Suppose the raw data are

$$\begin{aligned} \sigma^r(v_1) = 0.80, \quad \sigma^{r,c}(v_1) = 0.30; \quad \sigma^r(v_2) = 0.60, \quad \sigma^{r,c}(v_2) = 0.20; \quad \sigma^r(v_3) = 0.90, \quad \sigma^{r,c}(v_3) = 0.40; \\ \mu^r(e_1) = 0.70, \quad \nu^r(e_1) = 0.50. \end{aligned}$$

Step 1 (vertex normalization). For each v_i , compute $s_i = \sigma^r(v_i) + \sigma^{r,c}(v_i)$. If $s_i > 1$, set $\sigma(v_i) = \sigma^r(v_i)/s_i$ and $\sigma^c(v_i) = \sigma^{r,c}(v_i)/s_i$; otherwise keep the raw values. Explicitly:

$$\begin{aligned} s_1 = 0.80 + 0.30 = 1.10, \quad \sigma(v_1) = \frac{0.80}{1.10} = \frac{8}{11} \approx 0.727, \quad \sigma^c(v_1) = \frac{0.30}{1.10} = \frac{3}{11} \approx 0.273; \\ s_2 = 0.60 + 0.20 = 0.80 \leq 1, \quad \sigma(v_2) = 0.60, \quad \sigma^c(v_2) = 0.20; \\ s_3 = 0.90 + 0.40 = 1.30, \quad \sigma(v_3) = \frac{0.90}{1.30} = \frac{9}{13} \approx 0.692, \quad \sigma^c(v_3) = \frac{0.40}{1.30} = \frac{4}{13} \approx 0.308. \end{aligned}$$

Step 2 (edge adjustment). First compute the vertex-based minima:

$$\min_{v \in e_1} \sigma(v) = \min\left(\frac{8}{11}, 0.60\right) = 0.60, \quad \min_{v \in e_1} \sigma^c(v) = \min\left(\frac{3}{11}, 0.20\right) = 0.20.$$

Then

$$\mu(e_1) = \min\{0.70, 0.60\} = 0.60, \quad \nu(e_1) = \min\{0.50, 0.20\} = 0.20.$$

Since $\mu(e_1) + \nu(e_1) = 0.60 + 0.20 = 0.80 \leq 1$, the final rescaling step is not triggered.

Result. The output $\mathcal{H} = (V, E, \sigma, \sigma^c, \mu, \nu)$ satisfies $0 \leq \sigma(v_i), \sigma^c(v_i) \leq 1$ with $\sigma(v_i) + \sigma^c(v_i) \leq 1$ for $i = 1, 2, 3$; moreover $\mu(e_1) \leq \min_{v \in e_1} \sigma(v)$, $\nu(e_1) \leq \min_{v \in e_1} \sigma^c(v)$, and $0 \leq \mu(e_1) + \nu(e_1) \leq 1$. Hence $\mathcal{H}$ is an intuitionistic fuzzy 2-superhypergraph.

Theorem 25 (Correctness of Algorithm 1). *Let $\text{SuHyG}^{(n)} = (V, E)$ be any finite crisp n -superhypergraph and let $\sigma^r, \sigma^{r,c}: V \rightarrow [0, 1]$, $\mu^r, \nu^r: E \rightarrow [0, 1]$ be arbitrary raw functions. Algorithm 1 outputs $\mathcal{H} = (V, E, \sigma, \sigma^c, \mu, \nu)$ satisfying, for every $v \in V$ and $e \in E$,*

$$0 \leq \sigma(v) + \sigma^c(v) \leq 1, \quad 0 \leq \mu(e) + \nu(e) \leq 1, \quad \mu(e) \leq \min_{v \in e} \sigma(v), \quad \nu(e) \leq \min_{v \in e} \sigma^c(v).$$

In particular, $\mathcal{H}$ is an intuitionistic fuzzy n -superhypergraph in the sense of Definition 13.

Proof: First consider a vertex $v \in V$. If $s := \sigma^r(v) + \sigma^{r,c}(v) \leq 1$, the algorithm leaves $\sigma(v) = \sigma^r(v)$, $\sigma^c(v) = \sigma^{r,c}(v)$, so

$$0 \leq \sigma(v), \sigma^c(v) \leq 1, \quad \sigma(v) + \sigma^c(v) = s \leq 1.$$

If $s > 1$, then the algorithm sets $\sigma(v) = \sigma^r(v)/s$ and $\sigma^c(v) = \sigma^{r,c}(v)/s$, so

$$\sigma(v), \sigma^c(v) \geq 0, \quad \sigma(v), \sigma^c(v) \leq \frac{1}{s} \leq 1, \quad \sigma(v) + \sigma^c(v) = \frac{\sigma^r(v) + \sigma^{r,c}(v)}{s} = 1.$$

Thus in all cases $0 \leq \sigma(v) + \sigma^c(v) \leq 1$.

Now fix an edge $e \in E$. After the two min-steps we have

$$\mu(e) \leq \mu^r(e) \leq 1, \quad \mu(e) \leq \min_{v \in e} \sigma(v),$$

$$\nu(e) \leq \nu^r(e) \leq 1, \quad \nu(e) \leq \min_{v \in e} \sigma^c(v),$$

so both appurtenance inequalities hold and $0 \leq \mu(e), \nu(e) \leq 1$. If $t := \mu(e) + \nu(e) \leq 1$, nothing further is done and the sum constraint holds. If $t > 1$, the algorithm replaces $\mu(e), \nu(e)$ by $\mu(e)/t, \nu(e)/t$, so

$$\mu(e) + \nu(e) = \frac{\mu(e) + \nu(e)}{t} = 1, \quad 0 \leq \mu(e), \nu(e) \leq \frac{1}{t} \leq 1.$$

Moreover, since $t \geq 1$,

$$\frac{\mu(e)}{t} \leq \mu(e) \leq \min_{v \in e} \sigma(v), \quad \frac{\nu(e)}{t} \leq \nu(e) \leq \min_{v \in e} \sigma^c(v),$$

so appurtenance is preserved by the rescaling. Hence all four constraints in Definition 13 hold for every $e \in E$, and the output is an intuitionistic fuzzy n -superhypergraph. $\square$

Theorem 26 (Time and Space Complexity). *Assume the n -superhypergraph $\text{SuHyG}^{(n)} = (V, E)$ is represented by an incidence structure listing, for each edge $e \in E$, the vertices in e . Let*

$$I = \sum_{e \in E} |e|$$

be the total number of vertex-edge incidences, and let $d = \max_{e \in E} |e|$ be the maximum edge size. Then Algorithm 1 runs in time

$$O(|V|) \quad (\text{vertex normalization}) \quad + \quad O(I) \quad (\text{edge adjustment}) = O(|V| + I) = O(|V| + |E| \cdot d).$$

It uses $O(|V|)$ space for σ, σ^c and $O(|E|)$ for μ, ν , i.e. $O(|V| + |E|)$ additional space beyond the input incidence structure.

Proof: The vertex loop visits each $v \in V$ once, performing a constant amount of work, so it costs $O(|V|)$. The edge loop visits each $e \in E$ once and computes $\min_{v \in e} \sigma(v)$ and $\min_{v \in e} \sigma^c(v)$ by scanning the vertices of e , requiring $O(|e|)$ time per edge. Hence the total cost of the edge loop is $\sum_{e \in E} O(|e|) = O(I) = O(|E| d)$. The rescaling step per edge is $O(1)$.

The algorithm overwrites the given arrays $\sigma, \sigma^c, \mu, \nu$ in place and does not allocate any asymptotically larger data structure, so the extra memory required is $O(|V| + |E|)$. $\square$

5 | Decision-Making via Intuitionistic Fuzzy Superhypergraphs

We now describe a structured procedure for multi-criteria decision-making based on an intuitionistic fuzzy n -superhypergraph. We use a multi-level representation

$$\mathcal{H} = (V_0, \dots, V_n, E_1, \dots, E_n, \sigma, \sigma^c, \mu, \nu),$$

where:

- V_0 is the set of base alternatives.
- For each $k = 1, \dots, n$, the level- k superhyperedges $E_k \subseteq \mathcal{P}(V_{k-1})$ are nonempty subsets of V_{k-1} (hyperedges on level $k-1$ vertices).
- $\sigma, \sigma^c : V_0 \rightarrow [0, 1]$ are intuitionistic fuzzy membership and non-membership degrees assigned to base alternatives.
- For each $k = 1, \dots, n$, the functions $\mu, \nu : E_k \rightarrow [0, 1]$ give baseline intuitionistic fuzzy degrees for level- k superhyperedges. (For clarity, μ_k, ν_k below denote *propagated* degrees, while μ, ν are the fixed input functions.)
- For each $k = 1, \dots, n$, a weight distribution $\omega_k : E_k \rightarrow [0, 1]$ is given such that

$$\sum_{\substack{e \in E_k \\ v \in e}} \omega_k(e) = 1 \quad \text{for every } v \in V_{k-1},$$

i.e. for each vertex v at level $k-1$, the weights of all incident level- k superhyperedges form a convex combination.

Remark 27 (Outline of Key Steps). (1) *Initialization sets each base alternative's membership and non-membership to $(M_0(v), N_0(v)) = (\sigma(v), \sigma^c(v))$.*

(2) *At level k , each superhyperedge $e \in E_k$ receives propagated degrees*

$$\mu_k(e) = \mu(e) \min_{v \in e} M_{k-1}(v), \quad \nu_k(e) = \nu(e) \min_{v \in e} N_{k-1}(v),$$

followed, if necessary, by a normalization step ensuring $\mu_k(e) + \nu_k(e) \leq 1$.

(3) *The degrees μ_k, ν_k are then fused back into each $v \in V_{k-1}$ using the convex combination determined by the weights $\omega_k(e)$ for all incident edges $e \ni v$.*

(4) *After n levels, each alternative $a \in A \subseteq V_0$ has a final pair $(M_n(a), N_n(a))$. The score $S(a) = M_n(a) - N_n(a) \in [-1, 1]$ balances membership and non-membership and is used to rank the alternatives.*

Example 28 (Supplier Selection). Consider a one-level intuitionistic fuzzy superhypergraph ($n = 1$) for selecting among three suppliers A, B, C . The base alternatives are

$$V_0 = \{A, B, C\},$$

and the level-1 superhyperedges represent two geographic clusters:

$$E_1 = \{e_{\text{dom}} = \{A, B\}, e_{\text{int}} = \{C\}\}.$$

Vertex degrees are

$$\sigma(A) = 0.80, \sigma^c(A) = 0.10; \quad \sigma(B) = 0.60, \sigma^c(B) = 0.20; \quad \sigma(C) = 0.70, \sigma^c(C) = 0.15,$$

and baseline edge degrees are

$$\mu(e_{\text{dom}}) = 0.90, \nu(e_{\text{dom}}) = 0.05; \quad \mu(e_{\text{int}}) = 0.70, \nu(e_{\text{int}}) = 0.10.$$

Each supplier belongs to exactly one cluster, so we set

$$\omega_1(e_{\text{dom}}) = 1 \text{ for } v \in \{A, B\}, \quad \omega_1(e_{\text{int}}) = 1 \text{ for } v = C.$$

We take $A = \{A, B, C\}$.

Algorithm 2: Decision-Making with an Intuitionistic Fuzzy n -Superhypergraph

Input: $\mathcal{H} = (V_0, \dots, V_n, E_1, \dots, E_n, \sigma, \sigma^c, \mu, \nu)$,
 set of alternatives $A \subseteq V_0$,
 weights $\omega_k : E_k \rightarrow [0, 1]$ with $\sum_{e \ni v} \omega_k(e) = 1$ for every $v \in V_{k-1}$, $k = 1, \dots, n$.
Output: Ranking of A by intuitionistic fuzzy score

```

1 Step 0: Initialization.
2 foreach  $v \in V_0$  do
3    $M_0(v) \leftarrow \sigma(v)$ ;
4    $N_0(v) \leftarrow \sigma^c(v)$ ;
5 for  $k \leftarrow 1$  to  $n$  do
6   (1) Propagate to level- $k$  superhyperedges.
7   foreach  $e \in E_k$  do
8      $\mu_k(e) \leftarrow \mu(e) \min_{v \in e} M_{k-1}(v)$ ;
9      $\nu_k(e) \leftarrow \nu(e) \min_{v \in e} N_{k-1}(v)$ ;
10    if  $\mu_k(e) + \nu_k(e) > 1$  then
11       $t \leftarrow \mu_k(e) + \nu_k(e)$ ;
12       $\mu_k(e) \leftarrow \mu_k(e)/t$ ;
13       $\nu_k(e) \leftarrow \nu_k(e)/t$ ;
14    (2) Aggregate back to level- $k-1$  vertices.
15    foreach  $v \in V_{k-1}$  do
16      
$$M_k(v) \leftarrow \sum_{\substack{e \in E_k \\ v \in e}} \omega_k(e) \mu_k(e), \quad N_k(v) \leftarrow \sum_{\substack{e \in E_k \\ v \in e}} \omega_k(e) \nu_k(e).$$

17 Step  $n+1$ : Scoring and ranking.
18 foreach  $a \in A$  do
19    $S(a) \leftarrow M_n(a) - N_n(a)$ ;
20 return  $A$  sorted in descending order of  $S(a)$ ;
```

Step 0 (initialization). For each $X \in \{A, B, C\}$,

$$M_0(X) = \sigma(X), \quad N_0(X) = \sigma^c(X),$$

so explicitly:

$$M_0(A) = 0.80, \quad N_0(A) = 0.10; \quad M_0(B) = 0.60, \quad N_0(B) = 0.20; \quad M_0(C) = 0.70, \quad N_0(C) = 0.15.$$

Step 1 (propagate to clusters). For the domestic cluster $e_{\text{dom}} = \{A, B\}$,

$$\min_{v \in e_{\text{dom}}} M_0(v) = \min\{0.80, 0.60\} = 0.60, \quad \min_{v \in e_{\text{dom}}} N_0(v) = \min\{0.10, 0.20\} = 0.10.$$

Hence

$$\mu_1(e_{\text{dom}}) = 0.90 \times 0.60 = 0.54, \quad \nu_1(e_{\text{dom}}) = 0.05 \times 0.10 = 0.005,$$

and $\mu_1(e_{\text{dom}}) + \nu_1(e_{\text{dom}}) = 0.545 \leq 1$, so no rescaling is needed.

For the international cluster $e_{\text{int}} = \{C\}$,

$$\min_{v \in e_{\text{int}}} M_0(v) = M_0(C) = 0.70, \quad \min_{v \in e_{\text{int}}} N_0(v) = N_0(C) = 0.15,$$

so

$$\mu_1(e_{\text{int}}) = 0.70 \times 0.70 = 0.49, \quad \nu_1(e_{\text{int}}) = 0.10 \times 0.15 = 0.015,$$

with $\mu_1(e_{\text{int}}) + \nu_1(e_{\text{int}}) = 0.505 \leq 1$.

Step 2 (aggregate back to suppliers). Using the weights and the fact that each supplier lies in exactly one cluster, we obtain

$$\begin{aligned} M_1(A) &= \mu_1(e_{\text{dom}}) = 0.54, & N_1(A) &= \nu_1(e_{\text{dom}}) = 0.005; \\ M_1(B) &= \mu_1(e_{\text{dom}}) = 0.54, & N_1(B) &= \nu_1(e_{\text{dom}}) = 0.005; \\ M_1(C) &= \mu_1(e_{\text{int}}) = 0.49, & N_1(C) &= \nu_1(e_{\text{int}}) = 0.015. \end{aligned}$$

Step 3 (scoring and ranking). For each supplier $X \in \{A, B, C\}$,

$$S(X) = M_1(X) - N_1(X),$$

so

$$S(A) = 0.54 - 0.005 = 0.535, \quad S(B) = 0.54 - 0.005 = 0.535, \quad S(C) = 0.49 - 0.015 = 0.475.$$

Thus the final ranking is

$$A = B \succ C.$$

Theorem 29 (Correctness). *Under the above assumptions, for each level $k = 0, 1, \dots, n$ and each vertex $v \in V_{k-1}$,*

$$0 \leq M_k(v), N_k(v) \leq 1, \quad M_k(v) + N_k(v) \leq 1,$$

and for each $k = 1, \dots, n$ and each edge $e \in E_k$,

$$0 \leq \mu_k(e), \nu_k(e) \leq 1, \quad \mu_k(e) + \nu_k(e) \leq 1.$$

Consequently, for every alternative $a \in A$,

$$-1 \leq S(a) = M_n(a) - N_n(a) \leq 1.$$

Proof: We proceed by induction on the level k .

Base case ($k = 0$). By definition, $M_0(v) = \sigma(v) \in [0, 1]$ and $N_0(v) = \sigma^c(v) \in [0, 1]$ for all $v \in V_0$, and the Atanassov constraint on (σ, σ^c) implies $M_0(v) + N_0(v) \leq 1$.

Induction step. Assume the statements hold at level $k - 1$, i.e. for all $v \in V_{k-1}$,

$$0 \leq M_{k-1}(v), N_{k-1}(v) \leq 1, \quad M_{k-1}(v) + N_{k-1}(v) \leq 1.$$

First consider an edge $e \in E_k$. By construction,

$$\mu_k(e) = \mu(e) \min_{v \in e} M_{k-1}(v), \quad \nu_k(e) = \nu(e) \min_{v \in e} N_{k-1}(v).$$

We have $\mu(e), \nu(e) \in [0, 1]$ by assumption and $0 \leq M_{k-1}(v), N_{k-1}(v) \leq 1$ by the induction hypothesis, so

$$0 \leq \min_{v \in e} M_{k-1}(v) \leq 1, \quad 0 \leq \min_{v \in e} N_{k-1}(v) \leq 1,$$

and hence

$$0 \leq \mu_k(e), \nu_k(e) \leq 1.$$

If $\mu_k(e) + \nu_k(e) \leq 1$, the algorithm leaves them unchanged, so the sum constraint already holds. If $\mu_k(e) + \nu_k(e) > 1$, the algorithm replaces them by

$$\mu_k(e) \leftarrow \frac{\mu_k(e)}{t}, \quad \nu_k(e) \leftarrow \frac{\nu_k(e)}{t}, \quad t := \mu_k(e) + \nu_k(e) > 1,$$

which yields

$$\mu_k(e) + \nu_k(e) = \frac{\mu_k(e) + \nu_k(e)}{t} = 1, \quad 0 \leq \frac{\mu_k(e)}{t}, \frac{\nu_k(e)}{t} \leq 1.$$

Thus in all cases $0 \leq \mu_k(e), \nu_k(e) \leq 1$ and $\mu_k(e) + \nu_k(e) \leq 1$.

Now consider a vertex $v \in V_{k-1}$. By definition,

$$M_k(v) = \sum_{\substack{e \in E_k \\ v \in e}} \omega_k(e) \mu_k(e), \quad N_k(v) = \sum_{\substack{e \in E_k \\ v \in e}} \omega_k(e) \nu_k(e),$$

with $\omega_k(e) \geq 0$ and $\sum_{e \ni v} \omega_k(e) = 1$. Since each $\mu_k(e)$ and $\nu_k(e)$ lies in $[0, 1]$, we have

$$0 \leq M_k(v) \leq \sum_{e \ni v} \omega_k(e) \cdot 1 = 1, \quad 0 \leq N_k(v) \leq \sum_{e \ni v} \omega_k(e) \cdot 1 = 1.$$

For the sum,

$$M_k(v) + N_k(v) = \sum_{e \ni v} \omega_k(e)(\mu_k(e) + \nu_k(e)) \leq \sum_{e \ni v} \omega_k(e) \cdot 1 = 1.$$

Thus the assertions hold at level k as well.

Finally, for any alternative $a \in A \subseteq V_0$, the inequalities

$$0 \leq M_n(a), N_n(a) \leq 1, \quad M_n(a) + N_n(a) \leq 1$$

imply $-1 \leq M_n(a) - N_n(a) \leq 1$, so $-1 \leq S(a) \leq 1$. □

Theorem 30 (Complexity). *Let*

$$|V| := \sum_{k=0}^{n-1} |V_k|, \quad |E| := \sum_{k=1}^n |E_k|, \quad d_k := \max_{e \in E_k} |e|, \quad d := \max_{1 \leq k \leq n} d_k.$$

Assume that for each level k , the incidence structure of E_k is available (for example, as adjacency lists mapping each $v \in V_{k-1}$ to the incident edges). Then Algorithm 2 runs in time

$$O\left(\sum_{k=1}^n (|E_k| \cdot d_k + |V_{k-1}|)\right) = O(n(|E|d + |V|)),$$

and uses

$$O\left(\sum_{k=0}^n |V_k| + \sum_{k=1}^n |E_k|\right) = O(|V| + |E|)$$

additional space.

Proof: Initialization at level 0 visits each $v \in V_0$ once and performs constant work, so costs $O(|V_0|)$.

At a fixed level k , computing $\mu_k(e)$ and $\nu_k(e)$ for each $e \in E_k$ requires scanning the vertices in e , which takes $O(|e|)$ time per edge. Thus the propagation step costs $\sum_{e \in E_k} O(|e|) = O(|E_k| d_k)$. The rescaling (if needed) is $O(1)$ per edge.

The aggregation step must evaluate, for each $v \in V_{k-1}$,

$$M_k(v) = \sum_{e \ni v} \omega_k(e) \mu_k(e), \quad N_k(v) = \sum_{e \ni v} \omega_k(e) \nu_k(e).$$

Using adjacency lists from vertices to incident edges, the total work over all vertices at level $k-1$ is again proportional to the number of incidences between V_{k-1} and E_k , i.e. $O(|E_k| d_k)$, plus $O(|V_{k-1}|)$ for initialization and summation bookkeeping. Hence level k costs $O(|E_k| d_k + |V_{k-1}|)$.

Summing over levels $k = 1, \dots, n$ yields

$$O\left(\sum_{k=1}^n (|E_k| d_k + |V_{k-1}|)\right) \subseteq O(n(|E|d + |V|)).$$

Regarding space, the algorithm stores the arrays M_k and N_k for vertices at each level and μ_k, ν_k for edges at each level; their total size is $\sum_{k=0}^{n-1} O(|V_k|) + \sum_{k=1}^n O(|E_k|) = O(|V| + |E|)$ in addition to the input incidence data. □

6 | Discussion and Comparison

While classical hypergraphs [3], intuitionistic fuzzy hypergraphs, superhypergraphs [27], and fuzzy superhypergraphs [28] each address aspects of uncertainty or higher-order connectivity, our *intuitionistic fuzzy superhypergraphs* combine these strengths and yield several key benefits:

- **Enhanced decision support.** They facilitate decision-making within extremely intricate network structures, going beyond what standard fuzzy superhypergraphs can handle.
- **Hierarchical modeling.** They natively represent nested, multi-scale relationships that ordinary hypergraphs are unable to encode.
- **Richer uncertainty representation.** By assigning both membership and non-membership degrees in accordance with Atanassov’s constraint, they capture uncertainty more precisely.
- **Well-behaved operations.** They admit natural intuitionistic-fuzzy operations—such as level-cuts, union, intersection, and complement—while maintaining full consistency of the intuitionistic framework.

7 | Conclusion and Future Work

7.1 | Conclusion

In this paper, we introduced the concept of an *intuitionistic fuzzy superhypergraph*, which enriches intuitionistic fuzzy graphs by embedding them within the superhypergraph framework. We also explored a graph-based algorithm for constructing this structure.

7.2 | Future Work

Future research will investigate the integration of Rough Sets [29] and Plithogenic Sets [13], as well as extensions to bidirected graphs [30]. We also anticipate conducting computational experiments and developing software tools to facilitate the practical construction and analysis of intuitionistic fuzzy superhypergraphs.

Funding

This research has no funding source.

Data Availability

The datasets generated and/or analyzed during the current study are not publicly available due to privacy-preserving constraints but are available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare that there is no conflict of interest in this research.

Ethical Approval

This article does not contain any studies involving human participants or animals conducted by any of the authors.

References

- [1] Reinhard Diestel. *Graph theory*. Springer (print edition); Reinhard Diestel (eBooks), 2024.
- [2] Yifan Feng, Haoxuan You, Zizhao Zhang, Rongrong Ji, and Yue Gao. Hypergraph neural networks. In *Proceedings of the AAAI conference on artificial intelligence*, volume 33, pages 3558–3565, 2019.
- [3] Alain Bretto. Hypergraph theory. *An introduction. Mathematical Engineering*. Cham: Springer, 1, 2013.
- [4] Florentin Smarandache. *Extension of HyperGraph to n-SuperHyperGraph and to Plithogenic n-SuperHyperGraph, and Extension of HyperAlgebra to n-ary (Classical-/Neutro-/Anti-) HyperAlgebra*. Infinite Study, 2020.
- [5] Takaaki Fujita and Florentin Smarandache. *HyperGraph and SuperHyperGraph Theory with Applications (II): Graph Property and Parameter*, volume II of *HyperGraph and SuperHyperGraph Theory with Applications*. Neutrosophic Science International Association (NSIA) Publishing House, 1.0 edition, 2026.

- [6] Takaaki Fujita and Florentin Smarandache. *HyperGraph and SuperHyperGraph Theory with Applications*. Neutrosophic Science International Association (NSIA) Publishing House, 2026.
- [7] Takaaki Fujita and Florentin Smarandache. *HyperGraph and SuperHyperGraph Theory with Applications (IV): Uncertain Graph Theory*, volume IV of *HyperGraph and SuperHyperGraph Theory with Applications*. Neutrosophic Science International Association (NSIA) Publishing House, 1.0 edition, 2026.
- [8] Takaaki Fujita and Florentin Smarandache. *HyperGraph and SuperHyperGraph Theory with Applications (III): Intersection Graph and Graph Labeling*, volume III of *HyperGraph and SuperHyperGraph Theory with Applications*. Neutrosophic Science International Association (NSIA) Publishing House, 2026.
- [9] Azriel Rosenfeld. Fuzzy graphs. In *Fuzzy sets and their applications to cognitive and decision processes*, pages 77–95. Elsevier, 1975.
- [10] Muhammad Akram, Bijan Davvaz, and Feng Feng. Intuitionistic fuzzy soft k-algebras. *Mathematics in Computer Science*, 7:353–365, 2013.
- [11] Florentin Smarandache. *Hyperuncertain, superuncertain, and superhyperuncertain sets/logics/probabilities/statistics*. Infinite Study, 2017.
- [12] Takaaki Fujita and Florentin Smarandache. *A Dynamic Survey of Fuzzy, Intuitionistic Fuzzy, Neutrosophic, , and Extensional Sets*. Neutrosophic Science International Association (NSIA), 2025.
- [13] Florentin Smarandache. *Plithogenic set, an extension of crisp, fuzzy, intuitionistic fuzzy, and neutrosophic sets-revisited*. Infinite study, 2018.
- [14] Muhammad Azeem, Humera Rashid, Muhammad Kamran Jamil, Selma Gütmen, and Erfan Babae Tirkolae. Plithogenic fuzzy graph: A study of fundamental properties and potential applications. *Journal of Dynamics and Games*, pages 0–0, 2024.
- [15] Hossein Rashmanlou, Sovan Samanta, Madhumangal Pal, and Rajab Ali Borzooei. Intuitionistic fuzzy graphs with categorical properties. *Fuzzy information and Engineering*, 7(3):317–334, 2015.
- [16] Said Broumi, Mohamed Talea, Assia Bakali, and Florentin Smarandache. Single valued neutrosophic graphs. *Journal of New theory*, (10):86–101, 2016.
- [17] R. Jahir Hussain and M. S. Afya Farhana. Fuzzy chromatic number of fuzzy soft cycle and complete fuzzy soft graphs. *AIP Conference Proceedings*, 2023.
- [18] Claude Berge. *Hypergraphs: combinatorics of finite sets*, volume 45. Elsevier, 1984.
- [19] Takaaki Fujita. Directed acyclic superhypergraphs (dash): A general framework for hierarchical dependency modeling. *Neutrosophic Knowledge*, 6:72–86, 2025.
- [20] Florentin Smarandache. *SuperHyperFunction, SuperHyperStructure, Neutrosophic SuperHyperFunction and Neutrosophic SuperHyperStructure: Current understanding and future directions*. Infinite Study, 2023.
- [21] Lotfi A Zadeh. Fuzzy sets. *Information and control*, 8(3):338–353, 1965.
- [22] Mahdi Farshi and Bijan Davvaz. Generalized fuzzy hypergraphs and hypergroupoids. *Filomat*, 30(9):2375–2387, 2016.
- [23] Sankar Sahoo and Madhumangal Pal. Product of intuitionistic fuzzy graphs and degree. *Journal of Intelligent & Fuzzy Systems*, 32(1):1059–1067, 2017.
- [24] M. G. Karunambigai, R. Parvathi, and R. Buvaneswari. Arc in intuitionistic fuzzy graphs. *Notes on Intuitionistic Fuzzy Sets*, 17:37–47, 2011.
- [25] Muhammad Akram and Wieslaw A. Dudek. Intuitionistic fuzzy hypergraphs with applications. *Inf. Sci.*, 218:182–193, 2013.
- [26] Rangasamy Parvathi, S. Thilagavathi, and M. G. Karunambigai. Operations on intuitionistic fuzzy hypergraphs. *International Journal of Computer Applications*, 51:46–54, 2012.
- [27] Florentin Smarandache. *Introduction to the n-SuperHyperGraph-the most general form of graph today*. Infinite Study, 2022.
- [28] Mohammad Hamidi, Florentin Smarandache, and Mohadeseh Taghinezhad. *Decision Making Based on Valued Fuzzy Superhypergraphs*. Infinite Study, 2023.
- [29] Zdzisław Pawlak. Rough sets. *International journal of computer & information sciences*, 11:341–356, 1982.
- [30] Nanao Kita. Bidirected graphs i: Signed general kotzig-lovász decomposition. *arXiv: Combinatorics*, 2017.

Disclaimer/Publisher’s Note: The perspectives, opinions, and data shared in all publications are the sole responsibility of the individual authors and contributors, and do not necessarily reflect the views of Sciences Force or the editorial team. Sciences Force and the editorial team disclaim any liability for potential harm to individuals or property resulting from the ideas, methods, instructions, or products referenced in the content.