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Some Graph Structures in Fuzzy, Neutrosophic, and Plithogenic Graphs

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Abstract

In order to address the uncertainties inherent in real-world systems, various uncertain graph models—such as Fuzzy Graphs, Neutrosophic Graphs, Turiyam Neutrosophic Graphs, and Plithogenic Graphs—have been studied extensively. Similar to classical graph theory, a wide range of graph classes and parameters have also been examined in these frameworks, with applications in system operations, engineering, and beyond. In this paper, we introduce and analyze concepts such as Range Labeling, Cycle Connectivity, Linear Forest, Product Graph, Median Graph, and the Hyper-Zagreb Index within the contexts of Fuzzy, Neutrosophic, and Turiyam Neutrosophic Graphs.

Keywords: Neutrosophic graph, Fuzzy graph, Plithogenic Graph, Graph Class

1 | Introduction

1.1 | Uncertain Graph Theory

Graph theory investigates structures consisting of vertices and the edges that connect them, offering a versatile mathematical framework for analysing connectivity, flow, and structural properties in a wide variety of networks. It underpins modern applications ranging from graph neural networks [1] to complex engineering systems [2], and a rich literature has evolved around the study of specialised graph classes.

Classical graph theory, however, presumes precise knowledge of vertices and edges. Many real-world networks involve imprecision or ambiguity, motivating the development of *uncertain graph models*. This paper considers four such formalisms—*Fuzzy* [3], *Neutrosophic* [4], and *Plithogenic Graphs* [5]. Each extends the classical setting by embedding the underlying vertices and/or edges in an uncertainty-handling set theory: Fuzzy Sets [6], Neutrosophic Sets [7], or Plithogenic Sets [8]. For example, a Neutrosophic Graph annotates every vertex and edge with three independent membership degrees—truth, indeterminacy, and falsity—thereby capturing nuanced

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and sometimes contradictory information. By accommodating graded uncertainty, these models offer flexible tools for tackling practical problems that lie beyond the reach of traditional graph theory.

1.2 | Our Contribution

This work enriches the uncertain-graph literature by introducing and analysing six advanced notions—*Range Labeling*, *Cycle Connectivity*, *Linear Forests*, *Product Graphs*, *Median Graphs*, and the *Hyper-Zagreb Index*—within the frameworks of Fuzzy Graphs, Neutrosophic Graphs, and Turiyam Neutrosophic Graphs. We establish formal definitions, investigate their structural properties, and examine interrelations among these concepts, thereby deepening the theoretical foundations of uncertain graph theory and expanding its repertoire for real-world applications.

2 | Preliminaries

This section offers a concise overview of the definitions and notation employed throughout the paper. Unless explicitly stated otherwise, all graphs considered here are assumed to be finite.

2.1 | Fuzzy, Neutrosophic Graphs, and Turiyam Neutrosophic Graphs

In this subsection, we examine Fuzzy Graphs, Neutrosophic Graphs, and Turiyam Neutrosophic Graphs. Fuzzy graphs are frequently discussed in comparison with crisp graphs, which represent the classical form of graphs [9, 10].

Definition 1 (Crisp Graph). (cf.[9, 10]) A *crisp graph* is an ordered pair $G = (V, E)$, where:

- V is a finite, non-empty set of vertices.
- $E \subseteq V \times V$ is a set of edges, where each edge is an unordered pair of distinct vertices.

Formally, for any edge $(u, v) \in E$, the following holds:

$$(u, v) \in E \iff u \neq v \quad \text{and} \quad u, v \in V$$

This implies that there are no loops (i.e., no edges of the form (v, v)) and edges represent binary relationships between distinct vertices.

Considering the above context, we provide the definitions of Fuzzy, Intuitionistic Fuzzy, Neutrosophic, and Turiyam Neutrosophic Graphs below. These definitions have been streamlined for clarity and simplicity. It is worth noting that Turiyam Neutrosophic Graphs are generally known to be a generalization of Quadripartitioned Neutrosophic Graphs (cf.[11]).

Definition 2 (Unified Graphs Framework: Fuzzy, Intuitionistic Fuzzy, Neutrosophic, and Turiyam Neutrosophic Graphs). (cf.[12]) Let $G = (V, E)$ be a classical graph with a set of vertices V and a set of edges E . Depending on the type of graph, each vertex $v \in V$ and edge $e \in E$ is assigned membership values to represent various degrees of truth, indeterminacy, and falsity.

(1) *Fuzzy Graph* [9]:

- Each vertex $v \in V$ is assigned a membership degree $\sigma(v) \in [0, 1]$, representing the degree of participation of v in the fuzzy graph.
- Each edge $e = (u, v) \in E$ is assigned a membership degree $\mu(u, v) \in [0, 1]$, representing the strength of the connection between u and v .

(2) *Intuitionistic Fuzzy Graph (IFG)* [13]:

- Each vertex $v \in V$ is assigned two values: $\mu_A(v) \in [0, 1]$ (degree of membership) and $\nu_A(v) \in [0, 1]$ (degree of non-membership), such that $\mu_A(v) + \nu_A(v) \leq 1$.
- Each edge $e = (u, v) \in E$ is assigned two values: $\mu_B(u, v) \in [0, 1]$ (degree of membership) and $\nu_B(u, v) \in [0, 1]$ (degree of non-membership), such that $\mu_B(u, v) + \nu_B(u, v) \leq 1$.

(3) *Neutrosophic Graph* [14, 4]:

- Each vertex $v \in V$ is assigned a triple $\sigma(v) = (\sigma_T(v), \sigma_I(v), \sigma_F(v))$, where:
 - $\sigma_T(v) \in [0, 1]$ is the truth-membership degree,
 - $\sigma_I(v) \in [0, 1]$ is the indeterminacy-membership degree,
 - $\sigma_F(v) \in [0, 1]$ is the falsity-membership degree,
 - $\sigma_T(v) + \sigma_I(v) + \sigma_F(v) \leq 3$.
- Each edge $e = (u, v) \in E$ is assigned a triple $\mu(e) = (\mu_T(e), \mu_I(e), \mu_F(e))$, representing the truth, indeterminacy, and falsity degrees for the connection between u and v .

(4) *Turiyam Neutrosophic Graph* [15]:

- Each vertex $v \in V$ is assigned a quadruple $\sigma(v) = (t(v), iv(v), fv(v), lv(v))$, where:
 - $t(v) \in [0, 1]$ is the truth value,
 - $iv(v) \in [0, 1]$ is the indeterminacy value,
 - $fv(v) \in [0, 1]$ is the falsity value,
 - $lv(v) \in [0, 1]$ is the liberal state value,
 - $t(v) + iv(v) + fv(v) + lv(v) \leq 4$.
- Each edge $e = (u, v) \in E$ is similarly assigned a quadruple representing the same parameters for the connection between u and v .

Example 3 (Fuzzy Graph: Social Trust Network). *Scenario.* Consider two people A, B connected in a social trust network. Each vertex represents a person, and the edge weight represents trust.

Vertex memberships.

$$\sigma(A) = 0.9, \quad \sigma(B) = 0.8.$$

Edge membership.

$$\mu(A, B) = 0.7,$$

indicating a moderately strong trust relation. Thus, this forms a simple fuzzy graph where both nodes and edges have degrees of participation in $[0, 1]$.

Example 4 (Neutrosophic Graph: Communication Link Reliability). *Scenario.* Consider a communication link between two routers R_1, R_2 .

Vertex assignments.

$$\sigma(R_1) = (0.8, 0.1, 0.1), \quad \sigma(R_2) = (0.7, 0.2, 0.1).$$

Edge assignment.

$$\mu(R_1, R_2) = (0.6, 0.3, 0.2).$$

This indicates that the connection has truth degree 0.6, indeterminacy 0.3, and falsity 0.2. It models incomplete knowledge in network reliability evaluation.

Example 5 (Turiyam Neutrosophic Graph: Policy Decision Network). *Scenario.* Vertices represent policy alternatives P_1, P_2 , and edges represent compatibility. Each entity has four states: truth, indeterminacy, falsity, and liberal state.

Vertex values.

$$\sigma(P_1) = (0.7, 0.1, 0.1, 0.1), \quad \sigma(P_2) = (0.6, 0.2, 0.1, 0.1).$$

Edge values.

$$\mu(P_1, P_2) = (0.5, 0.2, 0.1, 0.2).$$

Here, the additional liberal component captures a flexible or “open” dimension in decision-making, extending the neutrosophic framework. Thus, this is a **Turiyam Neutrosophic Graph**.

2.2 | Plithogenic Graphs

Plithogenic Graphs have recently emerged as a generalization of Fuzzy Graphs and Turiyam Neutrosophic Graphs, extending the concept to represent Plithogenic Sets [8]. These graphs are a focus of ongoing research and development [16, 17]. The formal definition is presented below.

Definition 6 (Plithogenic Graph). [16, 18] Let $G = (V, E)$ be a crisp graph where V is the set of vertices and $E \subseteq V \times V$ is the set of edges. A *Plithogenic Graph* PG is defined as:

$$PG = (PM, PN)$$

where:

- (1) *Plithogenic Vertex Set* $PM = (M, l, Ml, adf, aCf)$:
 - $M \subseteq V$: This represents the collection of all vertices in the graph.
 - l : An attribute linked to the vertices, characterizing their specific properties or roles.
 - Ml : The set of all possible values that the vertex attribute l can assume.
 - $adf : M \times Ml \rightarrow [0, 1]^s$: A function known as the *Degree of Appurtenance Function (DAF)*, which assigns a membership value to each vertex-attribute pair within the graph.
 - $aCf : Ml \times Ml \rightarrow [0, 1]^t$: The *Degree of Contradiction Function (DCF)* for vertices, quantifying the level of contradiction between pairs of attribute values.
- (2) *Plithogenic Edge Set* $PN = (N, m, Nm, bdf, bCf)$:
 - $N \subseteq E$: Denotes the set of all edges in the graph.
 - m : An attribute assigned to edges, describing specific features or relationships of the connections.
 - Nm : The set of all possible values that the edge attribute m can take.
 - $bdf : N \times Nm \rightarrow [0, 1]^s$: A function termed the *Degree of Appurtenance Function (DAF)* for edges, mapping edge-attribute pairs to a membership degree.
 - $bCf : Nm \times Nm \rightarrow [0, 1]^t$: The *Degree of Contradiction Function (DCF)* for edges, representing the level of contradiction between pairs of edge attribute values.

The Plithogenic Graph PG must adhere to the following constraints:

- (1) *Edge Appurtenance Constraint*: For all $(x, p), (y, q) \in M \times Ml$:

$$bdf((xy), (p, q)) \leq \min\{adf(x, p), adf(y, q)\}$$

where $xy \in N$ represents an edge connecting vertices x and y , and $(p, q) \in Nm \times Nm$ are their associated attribute values.

- (2) *Contradiction Function Constraint*: For all $(p, q), (r, s) \in Nm \times Nm$:

$$bCf((p, q), (r, s)) \leq \min\{aCf(p, r), aCf(q, s)\}$$

- (3) *Reflexivity and Symmetry of Contradiction Functions*:

$$\begin{aligned} aCf(p, p) &= 0, & \forall p \in Ml, \\ aCf(p, q) &= aCf(q, p), & \forall p, q \in Ml, \\ bCf(p, p) &= 0, & \forall p \in Nm, \\ bCf(p, q) &= bCf(q, p), & \forall p, q \in Nm. \end{aligned}$$

Example 7 (Transportation Network with Two Attributes). *Scenario.* Consider a simple crisp graph with three vertices $V = \{A, B, C\}$ and edges $E = \{AB, BC\}$. Suppose vertices represent *bus stations*, and edges represent *bus routes*. Let there be $s = 2$ attributes for vertices and edges: (1) *capacity*, (2) *safety*.

Plithogenic Vertex Set.

$$adf(A) = (0.8, 0.6), \quad adf(B) = (0.7, 0.7), \quad adf(C) = (0.9, 0.5).$$

Here, for example, vertex A has high capacity (0.8) but moderate safety (0.6).

Plithogenic Edge Set.

$$bdf(AB) = (0.6, 0.5), \quad bdf(BC) = (0.7, 0.4).$$

The edge values represent bus route capacity and safety levels, constrained by the product condition:

$$b_i(uv) \leq \min\{a_i(u), a_i(v)\}, \quad \forall i.$$

For instance, for edge AB : $\min\{a_1(A), a_1(B)\} = \min\{0.8, 0.7\} = 0.7 \geq b_1(AB) = 0.6$, condition satisfied.

Contradiction Functions. Assume a simple contradiction degree:

$$aCf(p, q) = |p - q|, \quad bCf(p, q) = |p - q|,$$

which is reflexive and symmetric as required:

$$aCf(p, p) = 0, \quad bCf(p, p) = 0, \quad aCf(p, q) = aCf(q, p).$$

This defines a valid **Plithogenic Graph** $PG = (PM, PN)$. The vertex memberships reflect the multi-attribute status of stations, while the edge memberships encode the service quality of routes. The DCF quantifies attribute conflicts (e.g., between high capacity and low safety). This compact example demonstrates how plithogenic graphs extend classical graphs to multi-attribute and contradictory environments.

3 | Result in this paper

In this section, we present the results of this paper.

3.1 | Range Labeling

Graph labeling is one of the well-studied areas in graph theory [19, 20]. This paper focuses on a specific type of labeling called Range Labeling. Range Labeling is a bijective mapping of graph vertices to consecutive integers, where each edge label represents the absolute difference between the labels of its incident vertices, ensuring that all edge labels are distinct [21]. The extension of this concept to the fuzzy environment is referred to as Fuzzy Range Labeling[22]. The definitions of related concepts are provided below.

Definition 8 (Labeling of a Graph). (cf.[23, 24]) Let $G = (V, E)$ be a graph, where V is the set of vertices and $E \subseteq V \times V$ is the set of edges. A *labeling* of G is a function f that assigns labels to the elements of V , E , or both. Labeling can be classified into the following types:

- (1) **Vertex Labeling:** A labeling $f : V \rightarrow L$, where L is a set of labels, assigns a label $f(v) \in L$ to each vertex $v \in V$.
- (2) **Edge Labeling:** A labeling $f : E \rightarrow L$ assigns a label $f(e) \in L$ to each edge $e \in E$.
- (3) **Total Labeling:** A labeling $f : V \cup E \rightarrow L$ assigns a label $f(x) \in L$ to each $x \in V \cup E$, where L is the label set.

The function f may adhere to additional constraints depending on the type of labeling. For instance:

- If G is labeled such that adjacent vertices receive distinct labels, f is called a *vertex coloring*.
- If $f(e) = |f(u) - f(v)|$ for $e = \{u, v\}$, f is called a *graceful labeling*.
- If $f(v) + f(e)$ (for an incident vertex v and edge e) is constant across all edges, f is a *magic labeling*.

Definition 9 (Range Labeling). [21] Let $G = (V, E)$ be a graph with n vertices. A *Range Labeling* is a bijection $f : V \rightarrow \{1, 2, \dots, n\}$ such that for each edge $e = (v_k, v_{k+1}) \in E$, the edge label $f^*(e)$ is defined as:

$$f^*(e) = \max(f(v_k), f(v_{k+1})) - \min(f(v_k), f(v_{k+1})).$$

Each edge label $f^*(e)$ is distinct.

Definition 10 (Range Graph). A graph G is called a *Range Graph* if it admits a range labeling.

Example 11 (Range Graph). *Scenario.* Consider a path graph P_3 with vertices $V = \{A, B, C\}$ and edges $E = \{AB, BC\}$.

Range labeling. Assign labels $f(A) = 1, f(B) = 2, f(C) = 3$.

Edge labels.

$$f^*(AB) = |f(A) - f(B)| = |1 - 2| = 1, \quad f^*(BC) = |f(B) - f(C)| = |2 - 3| = 1.$$

Since the condition requires *distinct* edge labels, this labeling fails. Instead, relabel as $f(A) = 1, f(B) = 3, f(C) = 2$. Then:

$$f^*(AB) = |1 - 3| = 2, \quad f^*(BC) = |3 - 2| = 1.$$

Now edge labels are distinct, so P_3 with this labeling is a **Range Graph**.

Definition 12 (Fuzzy Labeling Graph). [22] Let $G = (V, E)$ be a graph. A *Fuzzy Labeling* is defined by two bijective functions:

- $\sigma : V \rightarrow [0, 1]$ (vertex membership function),
- $\mu : V \times V \rightarrow [0, 1]$ (edge membership function).

These functions satisfy the following conditions:

- The membership values $\sigma(v)$ and $\mu(u, v)$ are distinct for all vertices $v \in V$ and edges $(u, v) \in E$.
- For all $u, v \in V$, it holds that:

$$\mu(u, v) < \min(\sigma(u), \sigma(v)).$$

A graph that admits fuzzy labeling is called a *Fuzzy Labeling Graph*.

Definition 13 (Fuzzy Range Graph). [22] A *Fuzzy Range Graph* is a fuzzy graph that satisfies both fuzzy labeling and range labeling conditions, where all vertex and edge membership values are distinct. It admits *Fuzzy Range Labeling*, meaning the following conditions hold:

- The vertex membership function $\sigma(v)$ is distinct for all vertices $v \in V$.
- The edge membership function $\mu(u, v)$ is distinct for all edges $(u, v) \in E$.

Example 14 (Fuzzy Range Graph). *Scenario.* Take the same path graph P_3 . Each vertex now has a fuzzy membership degree.

Vertex memberships.

$$\sigma(A) = 0.8, \quad \sigma(B) = 1.0, \quad \sigma(C) = 0.6.$$

Fuzzy range labels. For edge AB :

$$f^*(AB) = |\sigma(A) - \sigma(B)| = |0.8 - 1.0| = 0.2.$$

For edge BC :

$$f^*(BC) = |\sigma(B) - \sigma(C)| = |1.0 - 0.6| = 0.4.$$

Since edge labels (0.2, 0.4) are distinct, this fuzzy assignment defines a valid **Fuzzy Range Graph**. It generalizes the crisp case by encoding vertex differences in terms of fuzzy membership degrees.

Theorem 15. *A Fuzzy Range Graph generalizes both Range Graphs and Fuzzy Graphs.*

Proof: We demonstrate that a Fuzzy Range Graph encompasses the definitions of both Range Graphs and Fuzzy Graphs.

1. Generalization of Range Graphs: Let $G = (V, E)$ be a Range Graph. By definition, there exists a range labeling $f : V \rightarrow \{1, 2, \dots, n\}$ such that the edge labels $f^*(e)$ are distinct for all edges $e \in E$.

To construct a Fuzzy Range Graph from G , we define:

- A vertex membership function $\sigma : V \rightarrow [0, 1]$, where $\sigma(v) = \frac{f(v)}{n}$. Since $f(v)$ is distinct for all $v \in V$, $\sigma(v)$ is also distinct.
- An edge membership function $\mu : E \rightarrow [0, 1]$, where $\mu(e) = \frac{f^*(e)}{\max(f^*(e))}$. Since $f^*(e)$ is distinct for all $e \in E$, $\mu(e)$ is also distinct.

This construction satisfies the fuzzy labeling conditions:

$$\mu(u, v) < \min(\sigma(u), \sigma(v)) \quad \forall (u, v) \in E,$$

and therefore, G qualifies as a Fuzzy Range Graph.

2. Generalization of Fuzzy Graphs: Let $G = (V, E)$ be a Fuzzy Graph. By definition, there exist functions $\sigma : V \rightarrow [0, 1]$ and $\mu : V \times V \rightarrow [0, 1]$ representing the membership degrees of vertices and edges, respectively.

To construct a Fuzzy Range Graph from G , we assign:

- $f(v) = \lfloor \sigma(v) \cdot n \rfloor$, where $f(v)$ maps vertices to integers in $\{1, 2, \dots, n\}$ for some large n . This ensures $f(v)$ is distinct if $\sigma(v)$ is distinct.
- $f^*(e) = \lfloor \mu(u, v) \cdot n \rfloor$ for $e = (u, v)$, ensuring $f^*(e)$ is distinct if $\mu(u, v)$ is distinct.

Thus, G satisfies the range labeling conditions and becomes a Fuzzy Range Graph. $\square$

Definition 16 (Neutrosophic Range Labeling). Let $G = (V, E)$ be a neutrosophic graph, where each vertex $v \in V$ is associated with a triple $\sigma(v) = (\sigma_T(v), \sigma_I(v), \sigma_F(v))$, representing the truth-membership, indeterminacy-membership, and falsity-membership degrees, respectively. Each edge $e = (u, v) \in E$ is assigned a triple $\mu(e) = (\mu_T(e), \mu_I(e), \mu_F(e))$.

A *Neutrosophic Range Labeling* is defined as a bijective mapping $f : V \rightarrow \{1, 2, \dots, n\}$ such that the edge label $f^*(e)$ for each edge $e = (u, v)$ is given by:

$$f^*(e) = \max(f(u), f(v)) - \min(f(u), f(v)).$$

This edge label $f^*(e)$ is extended to incorporate neutrosophic components:

$$f^*(e) = (\max(f_T(u), f_T(v)) - \min(f_T(u), f_T(v)), \max(f_I(u), f_I(v)) - \min(f_I(u), f_I(v)), \max(f_F(u), f_F(v)) - \min(f_F(u), f_F(v))),$$

where $f_T(u)$, $f_I(u)$, and $f_F(u)$ represent the truth, indeterminacy, and falsity components of vertex u .

The labeling is valid if all edge labels are distinct in each component.

Theorem 17. *A Neutrosophic Range Labeling Graph generalizes Fuzzy Range Graphs, Range Graphs, and Neutrosophic Graphs.*

Proof: To demonstrate this theorem, we show that the definitions of Fuzzy Range Graphs, Range Graphs, and Neutrosophic Graphs can be embedded into the framework of a Neutrosophic Range Labeling Graph.

1. Generalization of Range Graphs: Let $G = (V, E)$ be a Range Graph with a bijection $f : V \rightarrow \{1, 2, \dots, n\}$ such that the edge labels $f^*(e)$ are distinct for all edges $e \in E$.

In the case of a Neutrosophic Range Labeling Graph, let the truth-membership component $\sigma_T(v) = f(v)/n$ for each vertex $v \in V$. Define $f_T(u) = \sigma_T(u) \cdot n$ for every vertex u . The neutrosophic edge label reduces to:

$$f^*(e) = (\max(f_T(u), f_T(v)) - \min(f_T(u), f_T(v)), 0, 0),$$

indicating that the indeterminacy and falsity components are constant. Therefore, a Range Graph can be represented as a special case of a Neutrosophic Range Labeling Graph.

2. Generalization of Fuzzy Range Graphs: Let $G = (V, E)$ be a Fuzzy Range Graph, where $\sigma : V \rightarrow [0, 1]$ and $\mu : E \rightarrow [0, 1]$ are vertex and edge membership functions, respectively. Define $\sigma_T(v) = \sigma(v)$, $\sigma_I(v) = 0$, and $\sigma_F(v) = 0$ for each vertex $v \in V$. Similarly, set $\mu_T(e) = \mu(e)$, $\mu_I(e) = 0$, and $\mu_F(e) = 0$ for each edge $e \in E$.

The neutrosophic range label for an edge $e = (u, v)$ becomes:

$$f^*(e) = (\max(\sigma_T(u), \sigma_T(v)) - \min(\sigma_T(u), \sigma_T(v)), 0, 0),$$

matching the fuzzy range labeling conditions. Thus, a Fuzzy Range Graph is a subset of Neutrosophic Range Labeling Graphs.

3. Generalization of Neutrosophic Graphs: Let $G = (V, E)$ be a Neutrosophic Graph, where each vertex $v \in V$ and edge $e \in E$ have neutrosophic membership degrees $\sigma(v) = (\sigma_T(v), \sigma_I(v), \sigma_F(v))$ and $\mu(e) = (\mu_T(e), \mu_I(e), \mu_F(e))$, respectively. For any given neutrosophic range labeling f , define:

$$f_T(u) = \sigma_T(u), \quad f_I(u) = \sigma_I(u), \quad f_F(u) = \sigma_F(u).$$

The neutrosophic edge label becomes:

$$\begin{aligned} f^*(e) = & (\max(f_T(u), f_T(v)) - \min(f_T(u), f_T(v)), \\ & \max(f_I(u), f_I(v)) - \min(f_I(u), f_I(v)), \\ & \max(f_F(u), f_F(v)) - \min(f_F(u), f_F(v))), \end{aligned}$$

preserving the neutrosophic structure. Therefore, a Neutrosophic Graph is a specific instance of a Neutrosophic Range Labeling Graph where the labels directly correspond to the neutrosophic membership values.

By construction, a Neutrosophic Range Labeling Graph can represent the structures of Fuzzy Range Graphs, Range Graphs, and Neutrosophic Graphs. Hence, the Neutrosophic Range Labeling Graph generalizes all three. $\square$

Definition 18 (Turiyam Neutrosophic Range Labeling). Let $G = (V, E)$ be a Turiyam Neutrosophic graph, where each vertex $v \in V$ is associated with a quadruple $\sigma(v) = (t(v), iv(v), fv(v), lv(v))$, representing the truth, indeterminacy, falsity, and liberal state values, respectively. Each edge $e = (u, v) \in E$ is assigned a quadruple $\mu(e) = (\mu_T(e), \mu_I(e), \mu_F(e), \mu_L(e))$.

A *Turiyam Neutrosophic Range Labeling* is defined as a bijective mapping $f : V \rightarrow \{1, 2, \dots, n\}$ such that the edge label $f^*(e)$ for each edge $e = (u, v)$ is given by:

$$f^*(e) = \max(f(u), f(v)) - \min(f(u), f(v)).$$

This edge label is extended to the Turiyam Neutrosophic context:

$$\begin{aligned} f^*(e) = & (\max(t(u), t(v)) - \min(t(u), t(v)), \max(iv(u), iv(v)) - \min(iv(u), iv(v)), \\ & \max(fv(u), fv(v)) - \min(fv(u), fv(v)), \max(lv(u), lv(v)) - \min(lv(u), lv(v))), \end{aligned}$$

where $t(u)$, $iv(u)$, $fv(u)$, and $lv(u)$ represent the corresponding components of vertex u .

The labeling is valid if all edge labels are distinct in each component.

Theorem 19. A *Turiyam Neutrosophic Range Labeling Graph* generalizes *Neutrosophic Range Labeling Graphs*, *Fuzzy Range Graphs*, *Range Graphs*, and *Turiyam Neutrosophic Graphs*.

Proof: To establish the generalization, we demonstrate how each of the specified graph types can be represented as a special case of the Turiyam Neutrosophic Range Labeling Graph.

1. Generalization of Range Graphs: Let $G = (V, E)$ be a Range Graph with a bijective mapping $f : V \rightarrow \{1, 2, \dots, n\}$. In a Turiyam Neutrosophic Range Labeling Graph, we define the truth component $t(v) = f(v)/n$ for each vertex $v \in V$, and set $iv(v) = 0$, $fv(v) = 0$, and $lv(v) = 0$.

The edge label in this case becomes:

$$f^*(e) = (\max(t(u), t(v)) - \min(t(u), t(v)), 0, 0, 0),$$

where $t(u) = f(u)/n$. This reduction aligns with the definition of a Range Graph, showing it is a subset of Turiyam Neutrosophic Range Labeling Graphs.

2. Generalization of Fuzzy Range Graphs: Let $G = (V, E)$ be a Fuzzy Range Graph where $\sigma : V \rightarrow [0, 1]$ and $\mu : E \rightarrow [0, 1]$ represent the vertex and edge membership functions, respectively. In the Turiyam Neutrosophic Range Labeling Graph, set:

$$t(v) = \sigma(v), \quad iv(v) = 0, \quad fv(v) = 0, \quad lv(v) = 0,$$

and define the edge membership function accordingly:

$$f^*(e) = (\max(t(u), t(v)) - \min(t(u), t(v)), 0, 0, 0).$$

This satisfies the conditions for a Fuzzy Range Graph, indicating it is a special case of a Turiyam Neutrosophic Range Labeling Graph.

3. Generalization of Neutrosophic Range Labeling Graphs: Let $G = (V, E)$ be a Neutrosophic Range Labeling Graph where each vertex v is associated with a triple $\sigma(v) = (\sigma_T(v), \sigma_I(v), \sigma_F(v))$. In the Turiyam Neutrosophic Range Labeling Graph, set:

$$t(v) = \sigma_T(v), \quad iv(v) = \sigma_I(v), \quad fv(v) = \sigma_F(v), \quad lv(v) = 0.$$

The edge label becomes:

$$f^*(e) = (\max(t(u), t(v)) - \min(t(u), t(v)), \max(iv(u), iv(v)) - \min(iv(u), iv(v)), \\ \max(fv(u), fv(v)) - \min(fv(u), fv(v)), 0),$$

which matches the neutrosophic range labeling definition. Therefore, a Neutrosophic Range Labeling Graph is a subset of a Turiyam Neutrosophic Range Labeling Graph.

4. Generalization of Turiyam Neutrosophic Graphs: Let $G = (V, E)$ be a Turiyam Neutrosophic Graph where each vertex v is assigned a quadruple $\sigma(v) = (t(v), iv(v), fv(v), lv(v))$ and each edge e a quadruple $\mu(e) = (\mu_T(e), \mu_I(e), \mu_F(e), \mu_L(e))$. In the Turiyam Neutrosophic Range Labeling Graph, assign a range label $f : V \rightarrow \{1, 2, \dots, n\}$ such that:

$$f^*(e) = (\max(t(u), t(v)) - \min(t(u), t(v)), \max(iv(u), iv(v)) - \min(iv(u), iv(v)), \\ \max(fv(u), fv(v)) - \min(fv(u), fv(v)), \max(lv(u), lv(v)) - \min(lv(u), lv(v))).$$

This aligns with the definition of a Turiyam Neutrosophic Graph and satisfies the range labeling conditions.

By construction, a Turiyam Neutrosophic Range Labeling Graph can represent the structures of Neutrosophic Range Labeling Graphs, Fuzzy Range Graphs, Range Graphs, and Turiyam Neutrosophic Graphs. Hence, it generalizes all four graph types. $\square$

Definition 20 (Plithogenic Range Labeling). Let $PG = (PM, PN)$ be a plithogenic graph, where PM and PN represent the plithogenic vertex and edge sets, respectively. Each vertex $v \in PM$ is associated with a membership vector $adf(v) = (a_1(v), a_2(v), \dots, a_s(v))$, and each edge $e \in PN$ is associated with a membership vector $bdf(e) = (b_1(e), b_2(e), \dots, b_s(e))$. The contradiction degree between vertices and edges is defined by the Degree of Contradiction Function (DCF).

A *Plithogenic Range Labeling* is defined as a bijective mapping $f : PM \rightarrow \{1, 2, \dots, n\}$, such that the edge label $f^*(e)$ for each edge $e = (u, v)$ is computed as:

$$f^*(e) = \max(f(u), f(v)) - \min(f(u), f(v)),$$

and further extended to the plithogenic context:

$$f^*(e) = (\max(a_i(u), a_i(v)) - \min(a_i(u), a_i(v)))_{i=1}^s,$$

where $a_i(u)$ is the i -th component of the membership vector of vertex u .

The labeling is valid if all edge labels are distinct in each component.

Example 21 (Two-Attribute Range Labeling on a Simple Graph). *Scenario.* Consider a plithogenic path graph with vertices $A-B-C$. Let $s = 2$ attributes: (1) *capacity*, (2) *security*.

Vertex membership vectors.

$$adf(A) = (0.8, 0.5), \quad adf(B) = (0.6, 0.7), \quad adf(C) = (0.9, 0.4).$$

Edge range labels. For edge AB :

$$f^*(AB) = (\max(0.8, 0.6) - \min(0.8, 0.6), \max(0.5, 0.7) - \min(0.5, 0.7)) = (0.2, 0.2).$$

For edge BC :

$$f^*(BC) = (\max(0.6, 0.9) - \min(0.6, 0.9), \max(0.7, 0.4) - \min(0.7, 0.4)) = (0.3, 0.3).$$

Validity check. The edge labels are distinct in each component:

$$f^*(AB) = (0.2, 0.2) \neq f^*(BC) = (0.3, 0.3).$$

Thus this assignment yields a valid **Plithogenic Range Labeling**. It captures attribute-wise differences between vertex memberships, ensuring edge labels uniquely encode plithogenic range properties.

Theorem 22. *Plithogenic Range Labeling generalizes Neutrosophic Range Labeling, Turiyam Neutrosophic Range Labeling, and Fuzzy Range Labeling. Specifically, by choosing appropriate dimensions and membership functions, a Plithogenic Range Labeling reduces to these specific labelings.*

Proof: A Plithogenic Graph $PG = (PM, PN)$ has vertices PM with membership vectors

$$adf(v) = (a_1(v), a_2(v), \dots, a_s(v))$$

, where $s \geq 1$. The Plithogenic Range Labeling assigns to each edge $e = (u, v)$ an edge label:

$$f^*(e) = (\max(a_i(u), a_i(v)) - \min(a_i(u), a_i(v)))_{i=1}^s.$$

To recover the specific range labelings:

- (1) *Fuzzy Range Labeling:* Set $s = 1$ and $a_1(v) = \sigma(v)$, where $\sigma(v)$ is the fuzzy membership function. The edge label becomes:

$$f^*(e) = \max(\sigma(u), \sigma(v)) - \min(\sigma(u), \sigma(v)),$$

which matches the definition of Fuzzy Range Labeling.

- (2) *Neutrosophic Range Labeling:* Set $s = 3$ and $(a_1(v), a_2(v), a_3(v)) = (\sigma_T(v), \sigma_I(v), \sigma_F(v))$, the neutrosophic components. The edge label becomes:

$$f^*(e) = (\max(\sigma_T(u), \sigma_T(v)) - \min(\sigma_T(u), \sigma_T(v)), \max(\sigma_I(u), \sigma_I(v)) - \min(\sigma_I(u), \sigma_I(v)), \max(\sigma_F(u), \sigma_F(v)) - \min(\sigma_F(u), \sigma_F(v))),$$

matching the Neutrosophic Range Labeling.

- (3) *Turiyam Neutrosophic Range Labeling:* Set $s = 4$ and $(a_1(v), a_2(v), a_3(v), a_4(v)) = (t(v), iv(v), fv(v), lv(v))$, the Turiyam Neutrosophic components. The edge label becomes:

$$f^*(e) = (\max(t(u), t(v)) - \min(t(u), t(v)), \max(iv(u), iv(v)) - \min(iv(u), iv(v)), \max(fv(u), fv(v)) - \min(fv(u), fv(v)), \max(lv(u), lv(v)) - \min(lv(u), lv(v))),$$

which corresponds to the Turiyam Neutrosophic Range Labeling.

Thus, Plithogenic Range Labeling encompasses the other range labelings as special cases. $\square$

Theorem 23. *In a Plithogenic Range Labeling of a graph with n vertices, the maximum possible edge label in each component is $n - 1$.*

Proof: Since the vertex labels are bijectively mapped to the set $\{1, 2, \dots, n\}$, the maximum difference between any two vertex labels is $n - 1$. Therefore, for each component i , the maximum edge label is:

$$f_i^*(e) = \max(a_i(u), a_i(v)) - \min(a_i(u), a_i(v)) \leq n - 1.$$

□

Theorem 24. *The number of possible distinct edge labels in each component of a Plithogenic Range Labeling is at most $n - 1$.*

Proof: The possible differences between vertex labels range from 1 to $n - 1$ since the labels are integers from 1 to n and are distinct. Therefore, the maximum number of distinct edge labels in each component is $n - 1$. □

Theorem 25. *In Plithogenic Range Labeling, the sum of all edge labels in each component is constant and depends only on the vertex labels.*

Proof: Let $f : PM \rightarrow \{1, 2, \dots, n\}$ be the bijective mapping. For each component i , consider all unordered pairs $\{u, v\}$ of vertices connected by an edge. The sum of all edge labels in component i is:

$$S_i = \sum_{e=(u,v) \in PN} (\max(a_i(u), a_i(v)) - \min(a_i(u), a_i(v))).$$

Since the mapping f is fixed, and the differences depend only on the assigned labels, S_i is constant for a given labeling. □

Theorem 26. *If a graph G is connected, then it admits a Plithogenic Range Labeling.*

Proof: Since G is connected and has n vertices, we can assign labels $f(v)$ from 1 to n bijectively. The edge labels computed as differences between vertex labels will be distinct because the vertex labels are distinct and the graph is connected, ensuring that all necessary differences are realized. □

3.2 | Median graph

A Median Graph is a connected graph where, for any three vertices, there is a unique vertex that lies on the shortest paths between each pair of vertices [25]. A Fuzzy Median Graph is an extension of the median graph to the context of fuzzy graphs [26, 27]. The definition is provided below.

Definition 27 (Median Graph). [25] Let $G_1, G_2, \dots, G_k$ be a set of undirected, connected graphs, where each graph is represented as $G_i = (V_i, E_i)$ for $1 \leq i \leq k$. The *Median Graph* $G_M = (V_M, E_M)$ is defined as the graph that best represents the set $\{G_1, G_2, \dots, G_k\}$. It satisfies the following properties:

- $V_M \subseteq \bigcup_{i=1}^k V_i$, meaning the vertex set of the median graph is a subset of the union of the vertex sets of the given graphs.
- For each vertex $v \in V_M$, the total distance from v to the corresponding vertices in all graphs G_i is minimized. Mathematically, for any vertex $v \in V_M$:

$$\sum_{i=1}^k d_{G_i}(v, v_i) \leq \sum_{i=1}^k d_{G_i}(u, u_i), \quad \forall u \in \bigcup_{i=1}^k V_i,$$

where v_i and u_i are corresponding vertices in G_i , and $d_{G_i}(v, v_i)$ denotes the shortest path distance between v and v_i in graph G_i .

- The edge set E_M consists of edges that maintain connectivity while preserving the median property.

Example 28 (Median Graph). Consider three input graphs G_1, G_2, G_3 , each being a path of length 2 but with different vertex labels:

$$G_1 : A - B - C, \quad G_2 : A - C - B, \quad G_3 : B - A - C.$$

Each graph has the same structure: three vertices connected in a path. The vertex set union is $\{A, B, C\}$.

The best representative is the path graph $A - B - C$, since the total distance of vertex correspondences (minimizing across the three graphs) is smallest in this configuration.

Thus, the **Median Graph** G_M is the path $A - B - C$.

Definition 29 (Fuzzy Median Graph). [28] Let $F_1, F_2, \dots, F_k$ be a set of fuzzy graphs, where each fuzzy graph $F_i = (V_i, \sigma_i, \mu_i)$ is defined as:

- V_i is the set of vertices.
- $\sigma_i : V_i \rightarrow [0, 1]$ assigns a membership degree to each vertex, representing the degree of membership of the vertex in the fuzzy set.
- $\mu_i : V_i \times V_i \rightarrow [0, 1]$ assigns a membership degree to each edge, representing the degree of membership of the edge in the fuzzy relation.

The *Fuzzy Median Graph* $F_M = (V_M, \sigma_M, \mu_M)$ is defined as the fuzzy graph that best represents the set $\{F_1, F_2, \dots, F_k\}$ by satisfying the following conditions:

- $V_M \subseteq \bigcup_{i=1}^k V_i$, meaning the vertex set of the fuzzy median graph is a subset of the union of the vertex sets of the given fuzzy graphs.
- The vertex membership function $\sigma_M : V_M \rightarrow [0, 1]$ is defined such that, for each vertex $v \in V_M$:

$$\sigma_M(v) = \min_i \sigma_i(v_i),$$

where v_i is the corresponding vertex in F_i .

- The edge membership function $\mu_M : V_M \times V_M \rightarrow [0, 1]$ is defined such that, for each edge $(u, v) \in E_M$:

$$\mu_M(u, v) = \min_i \mu_i(u_i, v_i),$$

where u_i and v_i are the corresponding vertices in F_i .

Intuitively, the fuzzy median graph is the graph that best captures the information presented in all given fuzzy graphs, maintaining the central position within the fuzzy graph set.

Example 30 (Fuzzy Median Graph). *Scenario.* Consider two fuzzy graphs F_1, F_2 with the same vertices $V = \{X, Y\}$ and a single edge XY .

Graph F_1 .

$$\sigma_1(X) = 0.9, \quad \sigma_1(Y) = 0.8, \quad \mu_1(XY) = 0.7.$$

Graph F_2 .

$$\sigma_2(X) = 0.8, \quad \sigma_2(Y) = 0.7, \quad \mu_2(XY) = 0.6.$$

Median graph construction.

$$\begin{aligned} \sigma_M(X) &= \min\{0.9, 0.8\} = 0.8, & \sigma_M(Y) &= \min\{0.8, 0.7\} = 0.7, \\ \mu_M(XY) &= \min\{0.7, 0.6\} = 0.6. \end{aligned}$$

The **Fuzzy Median Graph** is $F_M = (\{X, Y\}, \{XY\})$ with:

$$\sigma_M(X) = 0.8, \quad \sigma_M(Y) = 0.7, \quad \mu_M(XY) = 0.6.$$

It represents the conservative (minimum-based) integration of the input fuzzy graphs.

Theorem 31. *The Fuzzy Median Graph is a generalization of the Median Graph.*

Proof: To establish that the Fuzzy Median Graph generalizes the Median Graph, we demonstrate that the Median Graph can be viewed as a specific instance of the Fuzzy Median Graph when the membership degrees for all vertices and edges are limited to binary values (0 or 1), representing crisp membership.

Under the condition that the membership degrees $\sigma_i(v)$ and $\mu_i(u, v)$ in each fuzzy graph F_i are constrained to binary values (0 or 1), the fuzzy graphs $F_1, F_2, \dots, F_k$ reduce to corresponding crisp graphs $G_1, G_2, \dots, G_k$. As a result:

- For each vertex $v \in V_M$, the aggregated vertex membership function $\sigma_M(v) = \min_i \sigma_i(v_i)$ evaluates to 1 if v appears in all graphs G_i ; otherwise, it evaluates to 0.
- For each edge $(u, v) \in E_M$, the aggregated edge membership function $\mu_M(u, v) = \min_i \mu_i(u_i, v_i)$ evaluates to 1 if (u, v) exists in all graphs G_i ; otherwise, it evaluates to 0.

This simplification corresponds directly to the definition of the Median Graph, in which vertices and edges are selected to minimize the total distance across the given set of graphs.

Thus, the Fuzzy Median Graph reduces to the Median Graph under the assumption of crisp membership, confirming that the Fuzzy Median Graph is a generalization of the Median Graph. $\square$

Definition 32 (Neutrosophic Median Graph). Let $N_1, N_2, \dots, N_k$ be a set of neutrosophic graphs, where each neutrosophic graph $N_i = (V_i, \sigma_i, \mu_i)$ is defined as:

- V_i is the set of vertices.
- $\sigma_i(v) = (\sigma_{iT}(v), \sigma_{iI}(v), \sigma_{iF}(v))$ assigns truth, indeterminacy, and falsity degrees to each vertex $v \in V_i$.
- $\mu_i(e) = (\mu_{iT}(e), \mu_{iI}(e), \mu_{iF}(e))$ assigns truth, indeterminacy, and falsity degrees to each edge $e \in E_i$.

The *Neutrosophic Median Graph* $N_M = (V_M, \sigma_M, \mu_M)$ is the neutrosophic graph that best represents the set $\{N_1, N_2, \dots, N_k\}$ and satisfies the following conditions:

- $V_M \subseteq \bigcup_{i=1}^k V_i$, i.e., the vertex set of the neutrosophic median graph is a subset of the union of the vertex sets of the given neutrosophic graphs.
- The vertex membership function $\sigma_M(v) = (\sigma_{MT}(v), \sigma_{MI}(v), \sigma_{MF}(v))$ is defined as:
$$\sigma_{MT}(v) = \min_i \sigma_{iT}(v), \quad \sigma_{MI}(v) = \min_i \sigma_{iI}(v), \quad \sigma_{MF}(v) = \min_i \sigma_{iF}(v),$$
where $v \in V_M$.
- The edge membership function $\mu_M(e) = (\mu_{MT}(e), \mu_{MI}(e), \mu_{MF}(e))$ is defined as:
$$\mu_{MT}(e) = \min_i \mu_{iT}(e), \quad \mu_{MI}(e) = \min_i \mu_{iI}(e), \quad \mu_{MF}(e) = \min_i \mu_{iF}(e),$$
where $e \in E_M$.

Example 33 (Neutrosophic Median Graph in Social Networks). Consider a social network where vertices represent individuals and edges represent friendships. Each edge is assigned a neutrosophic value (T, I, F) , where T is the degree of confirmed friendship (truth), I the uncertainty of the relation (indeterminacy), and F the degree of false or misleading connection. The median vertex of three individuals u, v, w is the person m that minimizes the neutrosophic distance

$$d_N(m, u) + d_N(m, v) + d_N(m, w),$$

where d_N accounts for (T, I, F) . For example, among three people A, B, C , the neutrosophic median may be person B if $d_N(B, A) + d_N(B, C)$ is smaller than for any other candidate.

Example 34 (Neutrosophic Median Graph in Transportation). In a transportation system, vertices represent stations and edges represent routes. Each route has a neutrosophic weight (T, I, F) , where T measures reliability, I represents uncertain delays, and F indicates failures. Given three stations x, y, z , the neutrosophic median

station m is the one minimizing the total neutrosophic distance to all three. For instance, if the routes from station M to x, y, z are highly reliable (T close to 1) and with low I, F , then M becomes the neutrosophic median.

Theorem 35. *The Neutrosophic Median Graph generalizes both the Fuzzy Median Graph and the Median Graph.*

Proof: To prove this theorem, we show that the Fuzzy Median Graph and the Median Graph are specific cases of the Neutrosophic Median Graph.

Case 1: Generalization of the Fuzzy Median Graph A Fuzzy Median Graph $F_M = (V_M, \sigma_M, \mu_M)$ is defined by vertex membership degrees $\sigma_M(v)$ and edge membership degrees $\mu_M(e)$ in the range $[0, 1]$. This corresponds to the Neutrosophic Median Graph when the indeterminacy and falsity components are set to zero. Specifically:

- Set $\sigma_{iT}(v) = \sigma_i(v)$, $\sigma_{iI}(v) = 0$, and $\sigma_{iF}(v) = 0$ for all vertices $v \in V_i$ in each neutrosophic graph N_i .
- Set $\mu_{iT}(e) = \mu_i(e)$, $\mu_{iI}(e) = 0$, and $\mu_{iF}(e) = 0$ for all edges $e \in E_i$ in each neutrosophic graph N_i .

Under these conditions, the Neutrosophic Median Graph $N_M = (V_M, \sigma_M, \mu_M)$ reduces to a Fuzzy Median Graph:

$$\sigma_M(v) = \min_i \sigma_{iT}(v), \quad \mu_M(e) = \min_i \mu_{iT}(e).$$

This matches the definition of the Fuzzy Median Graph, demonstrating that the Fuzzy Median Graph is a special case of the Neutrosophic Median Graph.

Case 2: Generalization of the Median Graph The Median Graph $G_M = (V_M, E_M)$ is a crisp graph with vertices and edges, where membership degrees are binary (0 or 1). This corresponds to the Neutrosophic Median Graph when the truth components are binary, and the indeterminacy and falsity components are set to zero:

- Set $\sigma_{iT}(v) = 1$ if v is present in V_i , and 0 otherwise. Set $\sigma_{iI}(v) = 0$ and $\sigma_{iF}(v) = 0$.
- Set $\mu_{iT}(e) = 1$ if e is present in E_i , and 0 otherwise. Set $\mu_{iI}(e) = 0$ and $\mu_{iF}(e) = 0$.

Under these conditions, the Neutrosophic Median Graph $N_M = (V_M, \sigma_M, \mu_M)$ reduces to:

$$\sigma_M(v) = \min_i \sigma_{iT}(v), \quad \mu_M(e) = \min_i \mu_{iT}(e).$$

This aligns with the definition of the Median Graph, where vertices and edges are chosen based on their presence in all graphs, confirming that the Median Graph is also a special case of the Neutrosophic Median Graph.

By covering the Fuzzy Median Graph and the Median Graph as special cases, the Neutrosophic Median Graph is shown to be a generalization of both. $\square$

Definition 36 (Turiyam Neutrosophic Median Graph). Let $T_1, T_2, \dots, T_k$ be a set of Turiyam Neutrosophic graphs, where each Turiyam Neutrosophic graph $T_i = (V_i, \sigma_i, \mu_i)$ is defined as:

- V_i is the set of vertices.
- $\sigma_i(v) = (t_i(v), iv_i(v), fv_i(v), lv_i(v))$ assigns truth, indeterminacy, falsity, and liberal state values to each vertex $v \in V_i$.
- $\mu_i(e) = (t_i(e), iv_i(e), fv_i(e), lv_i(e))$ assigns truth, indeterminacy, falsity, and liberal state values to each edge $e \in E_i$.

The *Turiyam Neutrosophic Median Graph* $T_M = (V_M, \sigma_M, \mu_M)$ is the Turiyam Neutrosophic graph that best represents the set $\{T_1, T_2, \dots, T_k\}$ and satisfies the following conditions:

- $V_M \subseteq \bigcup_{i=1}^k V_i$, meaning the vertex set of the Turiyam Neutrosophic median graph is a subset of the union of the vertex sets of the given Turiyam Neutrosophic graphs.
- The vertex membership function $\sigma_M(v) = (t_M(v), iv_M(v), fv_M(v), lv_M(v))$ is defined as:

$$t_M(v) = \min_i t_i(v), \quad iv_M(v) = \min_i iv_i(v), \quad fv_M(v) = \min_i fv_i(v), \quad lv_M(v) = \min_i lv_i(v),$$

where $v \in V_M$.

- The edge membership function $\mu_M(e) = (t_M(e), iv_M(e), fv_M(e), lv_M(e))$ is defined as:

$$t_M(e) = \min_i t_i(e), \quad iv_M(e) = \min_i iv_i(e), \quad fv_M(e) = \min_i fv_i(e), \quad lv_M(e) = \min_i lv_i(e),$$

where $e \in E_M$.

Theorem 37. *The Turiyam Neutrosophic Median Graph generalizes the Neutrosophic Median Graph, the Fuzzy Median Graph, and the Median Graph.*

Proof: To establish that the Turiyam Neutrosophic Median Graph generalizes the Neutrosophic Median Graph, the Fuzzy Median Graph, and the Median Graph, we demonstrate how each of these graphs can be viewed as a special case of the Turiyam Neutrosophic Median Graph.

Case 1: Generalization of the Neutrosophic Median Graph A Neutrosophic Median Graph $N_M = (V_M, \sigma_M, \mu_M)$ is defined by truth, indeterminacy, and falsity membership functions for vertices and edges. To represent N_M as a Turiyam Neutrosophic Median Graph, set the liberal state components to zero:

- For vertices: $lv_i(v) = 0$ for all $v \in V_i$ and all graphs T_i .
- For edges: $lv_i(e) = 0$ for all $e \in E_i$ and all graphs T_i .

Under these conditions, the Turiyam Neutrosophic Median Graph $T_M = (V_M, \sigma_M, \mu_M)$ reduces to:

$$\begin{aligned} t_M(v) &= \min_i t_i(v), & iv_M(v) &= \min_i iv_i(v), & fv_M(v) &= \min_i fv_i(v), \\ t_M(e) &= \min_i t_i(e), & iv_M(e) &= \min_i iv_i(e), & fv_M(e) &= \min_i fv_i(e), \end{aligned}$$

which matches the definition of the Neutrosophic Median Graph. Thus, the Neutrosophic Median Graph is a special case of the Turiyam Neutrosophic Median Graph.

Case 2: Generalization of the Fuzzy Median Graph A Fuzzy Median Graph $F_M = (V_M, \sigma_M, \mu_M)$ uses a single membership function for vertices and edges. To represent F_M as a Turiyam Neutrosophic Median Graph, set the indeterminacy, falsity, and liberal state components to zero:

- For vertices: $iv_i(v) = 0, fv_i(v) = 0, lv_i(v) = 0$ for all $v \in V_i$ and all graphs T_i .
- For edges: $iv_i(e) = 0, fv_i(e) = 0, lv_i(e) = 0$ for all $e \in E_i$ and all graphs T_i .

Under these conditions, the Turiyam Neutrosophic Median Graph $T_M = (V_M, \sigma_M, \mu_M)$ reduces to:

$$t_M(v) = \min_i t_i(v), \quad t_M(e) = \min_i t_i(e),$$

which matches the definition of the Fuzzy Median Graph. Hence, the Fuzzy Median Graph is a special case of the Turiyam Neutrosophic Median Graph.

Case 3: Generalization of the Median Graph The Median Graph $G_M = (V_M, E_M)$ is a crisp graph where membership degrees are binary (0 or 1). To represent G_M as a Turiyam Neutrosophic Median Graph, set the truth membership to binary values, and the indeterminacy, falsity, and liberal state components to zero:

- For vertices: $t_i(v) = 1$ if $v \in V_i$, and 0 otherwise. Set $iv_i(v) = 0, fv_i(v) = 0, lv_i(v) = 0$ for all $v \in V_i$.
- For edges: $t_i(e) = 1$ if $e \in E_i$, and 0 otherwise. Set $iv_i(e) = 0, fv_i(e) = 0, lv_i(e) = 0$ for all $e \in E_i$.

Under these conditions, the Turiyam Neutrosophic Median Graph $T_M = (V_M, \sigma_M, \mu_M)$ reduces to:

$$t_M(v) = \min_i t_i(v), \quad t_M(e) = \min_i t_i(e),$$

which matches the definition of the Median Graph. Thus, the Median Graph is a special case of the Turiyam Neutrosophic Median Graph.

By demonstrating that the Neutrosophic Median Graph, Fuzzy Median Graph, and Median Graph are special cases of the Turiyam Neutrosophic Median Graph, we establish that the Turiyam Neutrosophic Median Graph generalizes all three. $\square$

Definition 38 (Plithogenic Median Graph). Let $P_1, P_2, \dots, P_k$ be a set of plithogenic graphs, where each plithogenic graph $P_i = (V_i, adf_i, \mu_i)$ is defined as:

- V_i is the set of vertices.

- $adf_i(v) = (a_1(v), a_2(v), \dots, a_s(v))$ is the Degree of Appurtenance Function (DAF) for each vertex $v \in V_i$.
- $\mu_i(e) = (m_1(e), m_2(e), \dots, m_s(e))$ is the membership vector for each edge $e \in E_i$.

The *Plithogenic Median Graph* $P_M = (V_M, adf_M, \mu_M)$ is the plithogenic graph that best represents the set $\{P_1, P_2, \dots, P_k\}$ and satisfies the following conditions:

- $V_M \subseteq \bigcup_{i=1}^k V_i$, indicating the vertex set of the plithogenic median graph is a subset of the union of the vertex sets of the given plithogenic graphs.
- The vertex Degree of Appurtenance Function $adf_M(v) = (a_{M1}(v), a_{M2}(v), \dots, a_{Ms}(v))$ is defined as:

$$a_{Mj}(v) = \min_i a_{ij}(v), \quad \text{for all } j = 1, 2, \dots, s,$$

where $v \in V_M$.

- The edge membership function $\mu_M(e) = (m_{M1}(e), m_{M2}(e), \dots, m_{Ms}(e))$ is defined as:

$$m_{Mj}(e) = \min_i m_{ij}(e), \quad \text{for all } j = 1, 2, \dots, s,$$

where $e \in E_M$.

Example 39 (Median of Two Simple Plithogenic Graphs). *Scenario.* Suppose we have two plithogenic graphs P_1 and P_2 , each with the same vertex set $V = \{A, B\}$ and a single edge AB . Let $s = 2$ attributes: (1) *capacity*, (2) *security*.

Graph P_1 .

$$adf_1(A) = (0.9, 0.6), \quad adf_1(B) = (0.8, 0.7), \quad \mu_1(AB) = (0.7, 0.5).$$

Graph P_2 .

$$adf_2(A) = (0.8, 0.5), \quad adf_2(B) = (0.7, 0.6), \quad \mu_2(AB) = (0.6, 0.4).$$

Median vertex memberships. For vertex A :

$$a_{M1}(A) = \min\{0.9, 0.8\} = 0.8, \quad a_{M2}(A) = \min\{0.6, 0.5\} = 0.5.$$

So $adf_M(A) = (0.8, 0.5)$.

For vertex B :

$$a_{M1}(B) = \min\{0.8, 0.7\} = 0.7, \quad a_{M2}(B) = \min\{0.7, 0.6\} = 0.6.$$

So $adf_M(B) = (0.7, 0.6)$.

Median edge membership.

$$m_{M1}(AB) = \min\{0.7, 0.6\} = 0.6, \quad m_{M2}(AB) = \min\{0.5, 0.4\} = 0.4.$$

So $\mu_M(AB) = (0.6, 0.4)$.

Conclusion. The Plithogenic Median Graph P_M has the same structure as P_1, P_2 , but its vertex and edge membership vectors take the *minimum* values across graphs.

$$adf_M(A) = (0.8, 0.5), \quad adf_M(B) = (0.7, 0.6), \quad \mu_M(AB) = (0.6, 0.4).$$

This ensures that P_M represents the most conservative (lower-bound) evaluation of the system across all attributes.

Theorem 40. *The Plithogenic Median Graph generalizes the Neutrosophic Median Graph, Turiyam Neutrosophic Median Graph, Fuzzy Median Graph, and Classical Median Graph. Specifically, by choosing appropriate dimensions and components of the Degree of Appurtenance Function (DAF), a Plithogenic Median Graph reduces to these specific median graphs.*

Proof: A Plithogenic Graph $P_i = (V_i, \text{adf}_i, \mu_i)$ has vertices V_i with Degree of Appurtenance Functions (DAFs) $\text{adf}_i(v) = (a_{i1}(v), a_{i2}(v), \dots, a_{is}(v))$, where $s \geq 1$. The Plithogenic Median Graph $P_M = (V_M, \text{adf}_M, \mu_M)$ is defined with vertex DAFs:

$$a_{Mj}(v) = \min_i a_{ij}(v), \quad \text{for } j = 1, 2, \dots, s.$$

To recover the specific median graphs:

- (1) *Fuzzy Median Graph:* Set $s = 1$ and $a_{i1}(v) = \sigma_i(v)$, where $\sigma_i(v)$ is the fuzzy membership function of vertex v in fuzzy graph F_i . The vertex DAF in the median graph becomes:

$$a_{M1}(v) = \min_i \sigma_i(v),$$

which matches the definition of the Fuzzy Median Graph.

- (2) *Neutrosophic Median Graph:* Set $s = 3$ and $(a_{i1}(v), a_{i2}(v), a_{i3}(v)) = (\sigma_{iT}(v), \sigma_{iI}(v), \sigma_{iF}(v))$, the neutrosophic components of vertex v in N_i . Then, the vertex DAF in the median graph is:

$$a_{Mj}(v) = \min_i \sigma_{ij}(v), \quad \text{for } j = T, I, F,$$

which corresponds to the Neutrosophic Median Graph.

- (3) *Turiyam Neutrosophic Median Graph:* Set $s = 4$ and $(a_{i1}(v), a_{i2}(v), a_{i3}(v), a_{i4}(v)) = (t_i(v), iv_i(v), fv_i(v), lv_i(v))$, the Turiyam Neutrosophic components. The vertex DAF in the median graph becomes:

$$a_{Mj}(v) = \min_i a_{ij}(v), \quad \text{for } j = 1, 2, 3, 4,$$

matching the Turiyam Neutrosophic Median Graph.

- (4) *Classical Median Graph:* Set $s = 0$, implying that there are no DAF components, and the Plithogenic Median Graph reduces to the classical median graph defined on the vertex set V_M with edges E_M determined by minimizing distances as per the classical definition.

Thus, the Plithogenic Median Graph encompasses the other median graphs as special cases by selecting appropriate dimensions and components. $\square$

Theorem 41. *For any finite set of plithogenic graphs $\{P_1, P_2, \dots, P_k\}$, there exists a Plithogenic Median Graph P_M as defined.*

Proof: Since each P_i has a finite vertex set V_i and finite DAFs $\text{adf}_i(v)$ for each vertex $v \in V_i$, the union $\bigcup_{i=1}^k V_i$ is finite. The vertex set V_M is a subset of this union. The DAF for each vertex $v \in V_M$ is defined as $a_{Mj}(v) = \min_i a_{ij}(v)$. Since the minimum of finitely many real numbers exists, $a_{Mj}(v)$ is well-defined. Similarly for edges. Thus, P_M exists. $\square$

Theorem 42. *The median operation is associative; that is, for any grouping of graphs, the median remains the same:*

$$\text{Median}(P_1, P_2, P_3) = \text{Median}(\text{Median}(P_1, P_2), P_3).$$

Proof: The median operation is defined by taking the minimum of DAF components. The minimum function is associative, i.e.,

$$\min(\min(a, b), c) = \min(a, b, c).$$

Therefore, grouping the graphs in any manner results in the same median graph P_M . $\square$

Theorem 43. *If all the plithogenic graphs P_i are connected, then the Plithogenic Median Graph P_M is also connected under certain conditions.*

Proof: Since each P_i is connected, there exists a path between any two vertices in V_i . The DAF components in P_M are the minimum of those in P_i , so the edges in P_M correspond to edges present in all P_i . Therefore, P_M retains the connectivity where possible. However, if some edges have minimum DAF components of zero (or are absent in any P_i), those edges may not be present in P_M , potentially disconnecting P_M . But if the edges necessary to maintain connectivity have non-zero DAF components in all P_i , they will be present in P_M . Under the assumption that all necessary edges are present in all P_i , P_M remains connected. $\square$

3.3 | Product graphs

Product graphs have been widely studied across various fields, including uncertain graphs, due to their significance [29]. In uncertain theory, structures like Product Fuzzy Graphs have been explored [30]. The definitions are presented below.

Definition 44 (Product Fuzzy Graph). [30, 31] Let $G^* = (V, E)$ be a graph, and let $\sigma : V \rightarrow [0, 1]$ and $\mu : V \times V \rightarrow [0, 1]$ be a fuzzy subset and a fuzzy relation, respectively. A fuzzy graph $G = (\sigma, \mu)$ is called a *product fuzzy graph* if:

$$\mu(x, y) \leq \sigma(x) \cdot \sigma(y), \quad \forall x, y \in V.$$

Definition 45 (Product Intuitionistic Fuzzy Graph). [32] An intuitionistic fuzzy graph $G = (V, E, \mu_1, \gamma_1, \mu_2, \gamma_2)$ is called a *product intuitionistic fuzzy graph* if:

$$\mu_2(x, y) \leq \mu_1(x) \cdot \mu_1(y), \quad \gamma_2(x, y) \leq \gamma_1(x) \cdot \gamma_1(y), \quad \forall x, y \in V.$$

Theorem 46. A Product Intuitionistic Fuzzy Graph generalizes the concept of a Product Fuzzy Graph.

Proof: To establish that a Product Intuitionistic Fuzzy Graph generalizes a Product Fuzzy Graph, we demonstrate that the latter is a special case of the former when the non-membership functions γ_1 and γ_2 in the intuitionistic fuzzy graph are set to zero.

Case: Reduction to Product Fuzzy Graph Consider a Product Intuitionistic Fuzzy Graph $G = (V, E, \mu_1, \gamma_1, \mu_2, \gamma_2)$. Let the non-membership functions $\gamma_1 : V \rightarrow [0, 1]$ and $\gamma_2 : V \times V \rightarrow [0, 1]$ be defined as:

$$\gamma_1(x) = 0, \quad \gamma_2(x, y) = 0, \quad \forall x, y \in V.$$

Under this restriction, the intuitionistic fuzzy graph reduces to a fuzzy graph $G^* = (\mu_1, \mu_2)$, where:

- $\mu_1(x)$ represents the membership degree of vertex x ,
- $\mu_2(x, y)$ represents the membership degree of the edge (x, y) .

The product intuitionistic fuzzy condition:

$$\mu_2(x, y) \leq \mu_1(x) \cdot \mu_1(y), \quad \gamma_2(x, y) \leq \gamma_1(x) \cdot \gamma_1(y), \quad \forall x, y \in V,$$

reduces to:

$$\mu_2(x, y) \leq \mu_1(x) \cdot \mu_1(y), \quad \forall x, y \in V,$$

which is precisely the defining condition of a Product Fuzzy Graph.

When the non-membership functions γ_1 and γ_2 are set to zero, the Product Intuitionistic Fuzzy Graph becomes equivalent to a Product Fuzzy Graph. Hence, the Product Intuitionistic Fuzzy Graph generalizes the Product Fuzzy Graph. $\square$

Definition 47 (Product Neutrosophic Graph). Let $G = (V, E, \sigma, \mu)$ be a neutrosophic graph, where:

- Each vertex $v \in V$ is associated with a triple $\sigma(v) = (\sigma_T(v), \sigma_I(v), \sigma_F(v))$, representing the truth, indeterminacy, and falsity degrees, respectively.
- Each edge $e = (u, v) \in E$ is associated with a triple $\mu(e) = (\mu_T(e), \mu_I(e), \mu_F(e))$, representing the truth, indeterminacy, and falsity degrees for the edge.

The neutrosophic graph G is called a *Product Neutrosophic Graph* if, for all edges $e = (u, v) \in E$, the following conditions hold:

$$\mu_T(e) \leq \sigma_T(u) \cdot \sigma_T(v), \quad \mu_I(e) \leq \sigma_I(u) \cdot \sigma_I(v), \quad \mu_F(e) \leq \sigma_F(u) \cdot \sigma_F(v).$$

This definition ensures that the membership values of each edge do not exceed the product of the corresponding membership values of its incident vertices.

Example 48 (Product Neutrosophic Graph). *Scenario.* Consider a neutrosophic graph with two vertices A, B and one edge AB . Each vertex is associated with a triple (T, I, F) :

Vertex memberships.

$$\sigma(A) = (0.8, 0.2, 0.1), \quad \sigma(B) = (0.7, 0.3, 0.2).$$

Edge product bounds.

$$T(A) \cdot T(B) = 0.8 \cdot 0.7 = 0.56, \quad I(A) \cdot I(B) = 0.2 \cdot 0.3 = 0.06, \quad F(A) \cdot F(B) = 0.1 \cdot 0.2 = 0.02.$$

Edge assignment.

$$\mu(AB) = (0.5, 0.05, 0.02).$$

Verification. For each component:

$$0.5 \leq 0.56, \quad 0.05 \leq 0.06, \quad 0.02 \leq 0.02.$$

Thus, the condition is satisfied, and this is a valid **Product Neutrosophic Graph**.

Theorem 49. *A Product Neutrosophic Graph generalizes both the Product Intuitionistic Fuzzy Graph and the Product Fuzzy Graph.*

Proof: To prove the generalization, we show that both the Product Intuitionistic Fuzzy Graph and the Product Fuzzy Graph are special cases of the Product Neutrosophic Graph under certain constraints.

Case 1: Reduction to Product Intuitionistic Fuzzy Graph Consider a Product Neutrosophic Graph $G = (V, E, \sigma, \mu)$, where each vertex $v \in V$ and edge $e = (u, v)$ have membership values:

$$\sigma(v) = (\sigma_T(v), \sigma_I(v), \sigma_F(v)), \quad \mu(e) = (\mu_T(e), \mu_I(e), \mu_F(e)).$$

If we restrict the falsity components $\sigma_F(v)$ and $\mu_F(e)$ to 0, the neutrosophic graph reduces to an intuitionistic fuzzy graph $G' = (V, E, \mu_1, \gamma_1, \mu_2, \gamma_2)$, where:

$$\mu_1(v) = \sigma_T(v), \quad \gamma_1(v) = \sigma_I(v),$$

$$\mu_2(e) = \mu_T(e), \quad \gamma_2(e) = \mu_I(e).$$

Under this restriction, the product neutrosophic condition:

$$\mu_T(e) \leq \sigma_T(u) \cdot \sigma_T(v), \quad \mu_I(e) \leq \sigma_I(u) \cdot \sigma_I(v), \quad \mu_F(e) \leq \sigma_F(u) \cdot \sigma_F(v),$$

reduces to:

$$\mu_2(e) \leq \mu_1(u) \cdot \mu_1(v), \quad \gamma_2(e) \leq \gamma_1(u) \cdot \gamma_1(v),$$

which is the defining condition for a Product Intuitionistic Fuzzy Graph.

Case 2: Reduction to Product Fuzzy Graph Further, if we additionally constrain the indeterminacy components $\sigma_I(v)$ and $\mu_I(e)$ to 0, the intuitionistic fuzzy graph reduces to a fuzzy graph $G'' = (V, E, \mu_1, \mu_2)$, where:

$$\mu_1(v) = \sigma_T(v), \quad \mu_2(e) = \mu_T(e).$$

In this case, the product neutrosophic condition:

$$\mu_T(e) \leq \sigma_T(u) \cdot \sigma_T(v), \quad \mu_I(e) \leq \sigma_I(u) \cdot \sigma_I(v), \quad \mu_F(e) \leq \sigma_F(u) \cdot \sigma_F(v),$$

reduces to:

$$\mu_2(e) \leq \mu_1(u) \cdot \mu_1(v),$$

which matches the condition for a Product Fuzzy Graph. $\square$

Definition 50 (Product Turiyam Neutrosophic Graph). Let $G = (V, E, \sigma, \mu)$ be a Turiyam Neutrosophic graph, where:

- Each vertex $v \in V$ is associated with a quadruple $\sigma(v) = (t(v), iv(v), fv(v), lv(v))$, representing the truth, indeterminacy, falsity, and liberal state values, respectively.
- Each edge $e = (u, v) \in E$ is associated with a quadruple $\mu(e) = (t(e), iv(e), fv(e), lv(e))$, representing the truth, indeterminacy, falsity, and liberal state values for the edge.

The Turiyam Neutrosophic graph G is called a *Product Turiyam Neutrosophic Graph* if, for all edges $e = (u, v) \in E$, the following conditions hold:

$$t(e) \leq t(u) \cdot t(v), \quad iv(e) \leq iv(u) \cdot iv(v), \quad fv(e) \leq fv(u) \cdot fv(v), \quad lv(e) \leq lv(u) \cdot lv(v).$$

This definition ensures that the membership values of each edge do not exceed the product of the corresponding membership values of its incident vertices.

Example 51 (Product Turiyam Neutrosophic Graph). *Scenario.* Consider a Turiyam Neutrosophic graph with two vertices X, Y and one edge XY . Each vertex has a quadruple (t, iv, fv, lv) .

Vertex memberships.

$$\sigma(X) = (0.9, 0.3, 0.1, 0.5), \quad \sigma(Y) = (0.8, 0.4, 0.2, 0.6).$$

Edge product bounds.

$$t(X) \cdot t(Y) = 0.9 \cdot 0.8 = 0.72, \quad iv(X) \cdot iv(Y) = 0.3 \cdot 0.4 = 0.12,$$

$$fv(X) \cdot fv(Y) = 0.1 \cdot 0.2 = 0.02, \quad lv(X) \cdot lv(Y) = 0.5 \cdot 0.6 = 0.30.$$

Edge assignment.

$$\mu(XY) = (0.70, 0.10, 0.02, 0.25).$$

Thus, this defines a valid **Product Turiyam Neutrosophic Graph**. It extends the neutrosophic case with an additional liberal component.

Theorem 52. A *Product Turiyam Neutrosophic Graph* generalizes the *Product Neutrosophic Graph*, *Product Intuitionistic Fuzzy Graph*, and *Product Fuzzy Graph*.

Proof: To prove the generalization, we show that the Product Neutrosophic Graph, Product Intuitionistic Fuzzy Graph, and Product Fuzzy Graph are special cases of the Product Turiyam Neutrosophic Graph under specific constraints.

Case 1: Reduction to Product Neutrosophic Graph Consider a Product Turiyam Neutrosophic Graph $G = (V, E, \sigma, \mu)$, where:

$$\sigma(v) = (t(v), iv(v), fv(v), lv(v)), \quad \mu(e) = (t(e), iv(e), fv(e), lv(e)).$$

If we restrict the liberal state components $lv(v)$ and $lv(e)$ to 0, the graph reduces to a Product Neutrosophic Graph $G' = (V, E, \sigma', \mu')$, where:

$$\sigma'(v) = (t(v), iv(v), fv(v)), \quad \mu'(e) = (t(e), iv(e), fv(e)).$$

Under this restriction, the Product Turiyam Neutrosophic Graph conditions:

$$t(e) \leq t(u) \cdot t(v), \quad iv(e) \leq iv(u) \cdot iv(v), \quad fv(e) \leq fv(u) \cdot fv(v), \quad lv(e) \leq lv(u) \cdot lv(v),$$

reduce to:

$$t(e) \leq t(u) \cdot t(v), \quad iv(e) \leq iv(u) \cdot iv(v), \quad fv(e) \leq fv(u) \cdot fv(v),$$

which are the defining conditions of a Product Neutrosophic Graph.

Case 2: Reduction to Product Intuitionistic Fuzzy Graph Further, if we also restrict the falsity components $fv(v)$ and $fv(e)$ to 0, the graph reduces to a Product Intuitionistic Fuzzy Graph $G'' = (V, E, \mu_1, \gamma_1, \mu_2, \gamma_2)$, where:

$$\mu_1(v) = t(v), \quad \gamma_1(v) = iv(v), \quad \mu_2(e) = t(e), \quad \gamma_2(e) = iv(e).$$

Under these constraints, the Product Turiyam Neutrosophic Graph conditions reduce further to:

$$\mu_2(e) \leq \mu_1(u) \cdot \mu_1(v), \quad \gamma_2(e) \leq \gamma_1(u) \cdot \gamma_1(v),$$

which are the defining conditions of a Product Intuitionistic Fuzzy Graph.

Case 3: Reduction to Product Fuzzy Graph Finally, if we additionally restrict the indeterminacy components $iv(v)$ and $iv(e)$ to 0, the graph reduces to a Product Fuzzy Graph $G''' = (V, E, \sigma', \mu')$, where:

$$\sigma'(v) = t(v), \quad \mu'(e) = t(e).$$

In this case, the Product Turiyam Neutrosophic Graph conditions:

$$t(e) \leq t(u) \cdot t(v), \quad iv(e) \leq iv(u) \cdot iv(v), \quad fv(e) \leq fv(u) \cdot fv(v), \quad lv(e) \leq lv(u) \cdot lv(v),$$

reduce to:

$$\mu'(e) \leq \sigma'(u) \cdot \sigma'(v),$$

which matches the defining condition of a Product Fuzzy Graph.

By appropriately restricting the liberal state, falsity, and indeterminacy components in the Product Turiyam Neutrosophic Graph, it can be reduced to a Product Neutrosophic Graph, Product Intuitionistic Fuzzy Graph, or Product Fuzzy Graph. Therefore, the Product Turiyam Neutrosophic Graph generalizes all three graph types. $\square$

Definition 53 (Product Plithogenic Graph). Let $PG = (PM, PN)$ be a plithogenic graph, where:

- Each vertex $v \in PM$ has a membership vector $adf(v) = (a_1(v), a_2(v), \dots, a_s(v))$, representing the Degree of Appurtenance Function (DAF).
- Each edge $e \in PN$ has a membership vector $bdf(e) = (b_1(e), b_2(e), \dots, b_s(e))$, representing the Degree of Appurtenance for edges.

The plithogenic graph PG is called a *Product Plithogenic Graph* if, for all edges $e = (u, v) \in PN$, the following conditions hold for each component i :

$$b_i(e) \leq a_i(u) \cdot a_i(v), \quad \text{for all } i = 1, 2, \dots, s.$$

This definition ensures that the membership values of each edge in the plithogenic context do not exceed the product of the corresponding membership values of its incident vertices.

Example 54 (Two-Attribute Product Plithogenic Graph). *Scenario.* Consider a simple graph with two vertices A, B connected by one edge AB . Let there be $s = 2$ attributes: (1) *capacity*, (2) *security*.

Vertex memberships.

$$adf(A) = (0.8, 0.6), \quad adf(B) = (0.7, 0.5).$$

Product bounds for edge AB .

$$a_1(A) \cdot a_1(B) = 0.8 \cdot 0.7 = 0.56, \quad a_2(A) \cdot a_2(B) = 0.6 \cdot 0.5 = 0.30.$$

Edge membership assignment.

$$bdf(AB) = (0.50, 0.25).$$

Verification. For component 1: $b_1(AB) = 0.50 \leq 0.56$. For component 2: $b_2(AB) = 0.25 \leq 0.30$. Hence the condition

$$b_i(AB) \leq a_i(A) \cdot a_i(B), \quad i = 1, 2$$

is satisfied.

Conclusion. This graph qualifies as a **Product Plithogenic Graph**. The product condition reflects the principle that the strength of a connection cannot exceed the multiplicative joint compatibility of its incident vertices across all considered attributes.

Theorem 55. *The Product Plithogenic Graph generalizes the Product Neutrosophic Graph, Product Turiyam Neutrosophic Graph, and Product Fuzzy Graph. Specifically, by choosing appropriate dimensions and components of the Degree of Appurtenance Function (DAF), a Product Plithogenic Graph reduces to these specific product graphs.*

Proof: A Product Plithogenic Graph $PG = (PM, PN)$ has vertices PM with membership vectors $adf(v) = (a_1(v), a_2(v), \dots, a_s(v))$, where $s \geq 1$. The edge membership vectors $bdf(e) = (b_1(e), b_2(e), \dots, b_s(e))$ satisfy:

$$b_i(e) \leq a_i(u) \cdot a_i(v), \quad \text{for all } i = 1, 2, \dots, s, \quad \text{and } e = (u, v) \in PN.$$

To recover the specific product graphs:

(1) *Product Fuzzy Graph:*

Set $s = 1$ and let $a_1(v) = \sigma(v)$, the fuzzy membership function. Then, the edge condition becomes:

$$b_1(e) \leq \sigma(u) \cdot \sigma(v),$$

which matches the definition of a Product Fuzzy Graph.

(2) *Product Neutrosophic Graph:*

Set $s = 3$ and let $(a_1(v), a_2(v), a_3(v)) = (\sigma_T(v), \sigma_I(v), \sigma_F(v))$, the neutrosophic components. Then, the edge conditions become:

$$b_1(e) \leq \sigma_T(u) \cdot \sigma_T(v), \quad b_2(e) \leq \sigma_I(u) \cdot \sigma_I(v), \quad b_3(e) \leq \sigma_F(u) \cdot \sigma_F(v),$$

which correspond to the Product Neutrosophic Graph.

(3) *Product Turiyam Neutrosophic Graph:*

Set $s = 4$ and let $(a_1(v), a_2(v), a_3(v), a_4(v)) = (t(v), iv(v), fv(v), lv(v))$, the Turiyam Neutrosophic components. Then, the edge conditions become:

$$b_1(e) \leq t(u) \cdot t(v), \quad b_2(e) \leq iv(u) \cdot iv(v), \quad b_3(e) \leq fv(u) \cdot fv(v), \quad b_4(e) \leq lv(u) \cdot lv(v),$$

matching the definition of a Product Turiyam Neutrosophic Graph.

Thus, the Product Plithogenic Graph encompasses the other product graphs as special cases by selecting appropriate dimensions and components. $\square$

Theorem 56. *For any plithogenic graph $PG = (PM, PN)$, a corresponding Product Plithogenic Graph exists.*

Proof: Given a plithogenic graph $PG = (PM, PN)$ with vertex membership vectors $adf(v)$ and edge membership vectors $bdf(e)$, we can define a Product Plithogenic Graph by adjusting the edge membership vectors $bdf(e)$ to satisfy:

$$b_i(e) = \min \{b_i(e), a_i(u) \cdot a_i(v)\}, \quad \forall e = (u, v) \in PN, \quad i = 1, 2, \dots, s.$$

Since $a_i(u), a_i(v) \in [0, 1]$, their product is in $[0, 1]$, and $b_i(e)$ can be adjusted accordingly. Therefore, a Product Plithogenic Graph exists for any given plithogenic graph. $\square$

Theorem 57. *In a Product Plithogenic Graph, the edge membership values $b_i(e)$ are bounded above by the minimum of the incident vertex membership values for each component i .*

Proof: Since $a_i(u), a_i(v) \in [0, 1]$, their product $a_i(u) \cdot a_i(v) \leq \min\{a_i(u), a_i(v)\}$. Therefore:

$$b_i(e) \leq a_i(u) \cdot a_i(v) \leq \min\{a_i(u), a_i(v)\}.$$

$\square$

Theorem 58. *If a Product Plithogenic Graph is strong (i.e., $b_i(e) = a_i(u) \cdot a_i(v)$ for all edges e and components i), then the product of two such graphs is also strong.*

Proof: Let PG_1 and PG_2 be two strong Product Plithogenic Graphs with the same vertex set PM . For each edge $e = (u, v) \in PN$ and component i :

$$b_i^{(1)}(e) = a_i^{(1)}(u) \cdot a_i^{(1)}(v), \quad b_i^{(2)}(e) = a_i^{(2)}(u) \cdot a_i^{(2)}(v).$$

The product graph $PG = PG_1 \times PG_2$ has vertex membership $a_i(u) = a_i^{(1)}(u) \cdot a_i^{(2)}(u)$ and edge membership:

$$b_i(e) = b_i^{(1)}(e) \cdot b_i^{(2)}(e) = [a_i^{(1)}(u) \cdot a_i^{(1)}(v)] \cdot [a_i^{(2)}(u) \cdot a_i^{(2)}(v)] = a_i(u) \cdot a_i(v).$$

Therefore, the product graph is also strong. $\square$

Theorem 59. *The product of connected Product Plithogenic Graphs is connected.*

Proof: Let PG_1 and PG_2 be connected Product Plithogenic Graphs. Since both graphs are connected, there exist paths between any two vertices in each graph. The product graph $PG = PG_1 \times PG_2$ will have edges wherever there are edges in both PG_1 and PG_2 . Therefore, there exists a path between any two vertices in PG , and thus PG is connected. $\square$

Theorem 60. *In a Product Plithogenic Graph, the edge membership values are less than or equal to those in the original plithogenic graph.*

Proof: In constructing the Product Plithogenic Graph, the edge membership values $b_i(e)$ are adjusted to satisfy $b_i(e) \leq a_i(u) \cdot a_i(v)$. Since $a_i(u) \cdot a_i(v) \leq b_i(e)$ (original value) when the original edge membership exceeds the product of vertex memberships, the new $b_i(e)$ is less than or equal to the original $b_i(e)$. $\square$

Theorem 61. *The complete plithogenic graph PG_N with all vertex membership components equal to 1 acts as a neutral element under the product operation, i.e.,*

$$PG \times PG_N = PG_N \times PG = PG.$$

Proof: Let $PG = (PM, PN)$ be a Product Plithogenic Graph with vertex membership vectors

$$adf(v) = (a_1(v), a_2(v), \dots, a_s(v))$$

.

Define $PG_N = (PM_N, PN_N)$ where $PM_N = PM$ and for all $v \in PM_N$, $adf_{PG_N}(v) = (1, 1, \dots, 1)$. The edge membership vectors in PG_N are defined such that $b_i(e) = 1$ for all edges e and components i .

Under the product operation, the resulting graph $PG' = PG \times PG_N$ has vertex membership vectors:

$$adf_{PG'}(v) = adf_{PG}(v) \cdot adf_{PG_N}(v) = adf_{PG}(v) \cdot (1, 1, \dots, 1) = adf_{PG}(v).$$

Similarly, the edge membership vectors remain unchanged:

$$b'_i(e) = b_i(e) \cdot 1 = b_i(e).$$

Therefore, $PG \times PG_N = PG$.

The same argument applies for $PG_N \times PG$, so PG_N acts as a neutral element under the product operation. $\square$

Theorem 62. *In a Product Plithogenic Graph, the edge membership conditions are symmetric with respect to the incident vertices, i.e.,*

$$b_i(e) \leq a_i(u) \cdot a_i(v) = a_i(v) \cdot a_i(u), \quad \forall e = (u, v) \in PN.$$

Proof: Since real number multiplication is commutative, for each component i :

$$a_i(u) \cdot a_i(v) = a_i(v) \cdot a_i(u).$$

Thus, the condition for edge membership $b_i(e) \leq a_i(u) \cdot a_i(v)$ remains the same if u and v are swapped. Therefore, the edge membership conditions are symmetric with respect to the incident vertices. $\square$

Theorem 63. *All edge membership values in a Product Plithogenic Graph are non-negative, i.e.,*

$$b_i(e) \geq 0, \quad \forall e \in PN, \quad \forall i = 1, 2, \dots, s.$$

Proof: The vertex membership components $a_i(u), a_i(v) \in [0, 1]$, and the edge membership components satisfy $b_i(e) \leq a_i(u) \cdot a_i(v)$. Since the product of two non-negative real numbers is non-negative, and $b_i(e)$ is less than or equal to this product, it follows that $b_i(e) \geq 0$. $\square$

Theorem 64. *Any subgraph of a Product Plithogenic Graph is also a Product Plithogenic Graph under the induced membership functions.*

Proof: Let $PG = (PM, PN)$ be a Product Plithogenic Graph, and let $PG' = (PM', PN')$ be a subgraph with $PM' \subseteq PM$ and $PN' \subseteq PN$. The vertex membership vectors for PG' are the restrictions of those in PG to PM' , i.e., $adf_{PG'}(v) = adf_{PG}(v)$ for $v \in PM'$.

For each edge $e = (u, v) \in PN'$, the edge membership vectors are the same as in PG , and the condition $b_i(e) \leq a_i(u) \cdot a_i(v)$ still holds because $u, v \in PM' \subseteq PM$.

Therefore, PG' satisfies the conditions of a Product Plithogenic Graph. $\square$

3.4 | Uncertain Cycle connectivity

Cycle connectivity measures the strength of cycles connecting two vertices in a graph, indicating their level of interconnection [33]. Fuzzy Cycle Connectivity extends this concept to Fuzzy Graphs [34, 3]. The definitions, along with related concepts, are presented below.

Definition 65 (Cycle Evaluation of Fuzzy Graphs). [34, 3] Let $G = (V, E, \sigma, \mu)$ be a fuzzy graph, where:

- V is the set of vertices,
- $E \subseteq V \times V$ is the set of edges,
- $\sigma : V \rightarrow [0, 1]$ is the vertex membership function, and
- $\mu : E \rightarrow [0, 1]$ is the edge membership function.

For any two vertices $u, v \in V$, define a set $\theta(u, v)$, called the *cycle evaluation* of u and v , as follows:

$$\theta(u, v) = \{\alpha : \alpha \in (0, 1], \text{ where } \alpha \text{ is the strength of a strong cycle passing through both } u \text{ and } v\}.$$

If there is no strong cycle containing both u and v , then we set $\theta(u, v) = \emptyset$.

Definition 66 (Cycle Connectivity Between Two Vertices). Let $G = (V, E, \sigma, \mu)$ be a fuzzy graph, and let $u, v \in V$. The *cycle connectivity* between u and v , denoted by $CG_{u,v}$, is defined as:

$$CG_{u,v} = \max\{\alpha : \alpha \in \theta(u, v)\}.$$

If $\theta(u, v) = \emptyset$, then $CG_{u,v} = 0$.

Definition 67 (Cycle Connectivity of a Fuzzy Graph). The *cycle connectivity* of a fuzzy graph $G = (V, E, \sigma, \mu)$, denoted by $CC(G)$, is defined as:

$$CC(G) = \max\{CG_{u,v} : u, v \in V\}.$$

That is, the cycle connectivity of a fuzzy graph is the maximum cycle connectivity among all pairs of vertices in G .

Definition 68 (Cycle Connectivity in Neutrosophic Graph). Let $G = (V, E, \sigma, \mu)$ be a neutrosophic graph, where:

- Each vertex $v \in V$ is assigned a triple $\sigma(v) = (\sigma_T(v), \sigma_I(v), \sigma_F(v))$, representing truth, indeterminacy, and falsity membership degrees.
- Each edge $e = (u, v) \in E$ is assigned a triple $\mu(e) = (\mu_T(e), \mu_I(e), \mu_F(e))$, representing truth, indeterminacy, and falsity degrees.

For any two vertices $u, v \in V$, define a set $\theta(u, v)$, called the *cycle evaluation* of u and v , as follows:

$$\theta(u, v) = \{\alpha_T, \alpha_I, \alpha_F : \alpha_T, \alpha_I, \alpha_F \in (0, 1]\}$$

where these values represent the strength of a strong cycle passing through both u and v .

If there is no strong cycle containing both u and v , then $\theta(u, v) = \emptyset$.

The *cycle connectivity* between u and v in G , denoted by $CG_{u,v}$, is defined as:

$$CG_{u,v} = (\max\{\alpha_T : \alpha_T \in \theta_T(u, v)\}, \max\{\alpha_I : \alpha_I \in \theta_I(u, v)\}, \max\{\alpha_F : \alpha_F \in \theta_F(u, v)\}).$$

If $\theta(u, v) = \emptyset$, then $CG_{u,v} = (0, 0, 0)$.

The *cycle connectivity* of the neutrosophic graph G , denoted by $CC(G)$, is defined as:

$$CC(G) = \max\{CG_{u,v} : u, v \in V\}.$$

Theorem 69. *The Cycle Connectivity in a Neutrosophic Graph generalizes the Cycle Connectivity of Fuzzy Graphs.*

Proof: To demonstrate that the cycle connectivity in a neutrosophic graph generalizes the cycle connectivity in fuzzy graphs, we show that the latter is a special case of the former when the indeterminacy and falsity membership degrees are set to zero.

Let $G = (V, E, \sigma, \mu)$ be a neutrosophic graph, where:

$$\sigma(v) = (\sigma_T(v), \sigma_I(v), \sigma_F(v)), \quad \mu(e) = (\mu_T(e), \mu_I(e), \mu_F(e)).$$

Reduction to Fuzzy Graphs If we restrict the indeterminacy and falsity components of the vertex and edge membership functions to zero, i.e.,:

$$\sigma_I(v) = 0, \quad \sigma_F(v) = 0, \quad \mu_I(e) = 0, \quad \mu_F(e) = 0,$$

then the neutrosophic graph reduces to a fuzzy graph $G' = (V, E, \sigma', \mu')$, where:

$$\sigma'(v) = \sigma_T(v), \quad \mu'(e) = \mu_T(e).$$

Cycle Evaluation Reduction In the neutrosophic graph, the cycle evaluation set $\theta(u, v)$ is given by:

$$\theta(u, v) = \{(\alpha_T, \alpha_I, \alpha_F) : \alpha_T, \alpha_I, \alpha_F \in (0, 1] \text{ and represent strong cycles containing } u \text{ and } v\}.$$

Under the restriction $\alpha_I = 0$ and $\alpha_F = 0$, the cycle evaluation set reduces to:

$$\theta'(u, v) = \{\alpha_T : \alpha_T \in (0, 1] \text{ and represents a strong cycle containing } u \text{ and } v\},$$

which corresponds to the cycle evaluation of a fuzzy graph.

Cycle Connectivity Reduction The cycle connectivity between two vertices u, v in the neutrosophic graph is:

$$CG_{u,v} = (\max\{\alpha_T : \alpha_T \in \theta_T(u, v)\}, \max\{\alpha_I : \alpha_I \in \theta_I(u, v)\}, \max\{\alpha_F : \alpha_F \in \theta_F(u, v)\}).$$

Under the restrictions $\alpha_I = 0$ and $\alpha_F = 0$, the cycle connectivity reduces to:

$$CG'_{u,v} = \max\{\alpha_T : \alpha_T \in \theta'(u, v)\},$$

which matches the definition of cycle connectivity in fuzzy graphs.

Graph-Wide Cycle Connectivity Reduction The cycle connectivity of the neutrosophic graph is:

$$CC(G) = \max\{CG_{u,v} : u, v \in V\}.$$

Under the restrictions $\alpha_I = 0$ and $\alpha_F = 0$, this reduces to:

$$CC'(G) = \max\{CG'_{u,v} : u, v \in V\},$$

which corresponds to the cycle connectivity of a fuzzy graph.

By appropriately restricting the indeterminacy and falsity components in a neutrosophic graph, it reduces to a fuzzy graph. Therefore, the cycle connectivity in a neutrosophic graph generalizes the cycle connectivity in fuzzy graphs. $\square$

Definition 70 (Cycle Connectivity in Turiyam Neutrosophic Graph). Let $G = (V, E, \sigma, \mu)$ be a Turiyam Neutrosophic graph, where:

- Each vertex $v \in V$ is assigned a quadruple $\sigma(v) = (t(v), iv(v), fv(v), lv(v))$, representing truth, indeterminacy, falsity, and liberal state values.
- Each edge $e = (u, v) \in E$ is assigned a quadruple $\mu(e) = (t(e), iv(e), fv(e), lv(e))$, representing truth, indeterminacy, falsity, and liberal state degrees.

For any two vertices $u, v \in V$, define a set $\theta(u, v)$, called the *cycle evaluation* of u and v , as follows:

$$\theta(u, v) = \{\alpha_T, \alpha_I, \alpha_F, \alpha_L : \alpha_T, \alpha_I, \alpha_F, \alpha_L \in (0, 1]\}$$

where these values represent the strength of a strong cycle passing through both u and v .

If there is no strong cycle containing both u and v , then $\theta(u, v) = \emptyset$.

The *cycle connectivity* between u and v in G , denoted by $CG_{u,v}$, is defined as:

$$CG_{u,v} = (\max\{\alpha_T : \alpha_T \in \theta_T(u, v)\}, \max\{\alpha_I : \alpha_I \in \theta_I(u, v)\}, \max\{\alpha_F : \alpha_F \in \theta_F(u, v)\}, \max\{\alpha_L : \alpha_L \in \theta_L(u, v)\}).$$

If $\theta(u, v) = \emptyset$, then $CG_{u,v} = (0, 0, 0, 0)$.

The *cycle connectivity* of the Turiyam Neutrosophic graph G , denoted by $CC(G)$, is defined as:

$$CC(G) = \max\{CG_{u,v} : u, v \in V\}.$$

Theorem 71. *The Cycle Connectivity in a Turiyam Neutrosophic Graph generalizes both the Cycle Connectivity in Neutrosophic Graphs and the Cycle Connectivity of Fuzzy Graphs.*

Proof: To demonstrate that the cycle connectivity in a Turiyam Neutrosophic Graph generalizes the cycle connectivity in Neutrosophic Graphs and Fuzzy Graphs, we show that these are special cases of the Turiyam Neutrosophic framework when the liberal state (lv) or additional components are removed.

Let $G = (V, E, \sigma, \mu)$ be a Turiyam Neutrosophic graph, where:

$$\sigma(v) = (t(v), iv(v), fv(v), lv(v)), \quad \mu(e) = (t(e), iv(e), fv(e), lv(e)).$$

Reduction to Neutrosophic Graphs If the liberal state component (lv) for all vertices and edges is set to zero, i.e.,:

$$lv(v) = 0, \quad lv(e) = 0, \quad \forall v \in V, e \in E,$$

then the Turiyam Neutrosophic Graph reduces to a Neutrosophic Graph $G' = (V, E, \sigma', \mu')$, where:

$$\sigma'(v) = (t(v), iv(v), fv(v)), \quad \mu'(e) = (t(e), iv(e), fv(e)).$$

Under this restriction:

- The cycle evaluation set $\theta(u, v)$ reduces to:
 $\theta'(u, v) = \{(\alpha_T, \alpha_I, \alpha_F) : \alpha_T, \alpha_I, \alpha_F \in (0, 1] \text{ and represent strong cycles passing through } u \text{ and } v\}.$
- The cycle connectivity between vertices u and v reduces to:
 $CG'_{u,v} = (\max\{\alpha_T : \alpha_T \in \theta_T(u, v)\}, \max\{\alpha_I : \alpha_I \in \theta_I(u, v)\}, \max\{\alpha_F : \alpha_F \in \theta_F(u, v)\}),$
which is the cycle connectivity in a Neutrosophic Graph.
- The graph-wide cycle connectivity becomes:

$$CC'(G) = \max\{CG'_{u,v} : u, v \in V\}.$$

Reduction to Fuzzy Graphs If, in addition to setting $lv(v) = 0$ and $lv(e) = 0$, the indeterminacy and falsity components are also set to zero, i.e.:

$$iv(v) = 0, \quad fv(v) = 0, \quad iv(e) = 0, \quad fv(e) = 0, \quad \forall v \in V, e \in E,$$

then the Turiyam Neutrosophic Graph reduces to a Fuzzy Graph $G'' = (V, E, \sigma'', \mu'')$, where:

$$\sigma''(v) = t(v), \quad \mu''(e) = t(e).$$

Under this restriction:

- The cycle evaluation set $\theta(u, v)$ reduces further to:
 $\theta''(u, v) = \{\alpha_T : \alpha_T \in (0, 1] \text{ and represents strong cycles passing through } u \text{ and } v\}.$
- The cycle connectivity between vertices u and v reduces to:
 $CG''_{u,v} = \max\{\alpha_T : \alpha_T \in \theta_T(u, v)\},$
which is the cycle connectivity in a Fuzzy Graph.
- The graph-wide cycle connectivity becomes:

$$CC''(G) = \max\{CG''_{u,v} : u, v \in V\}.$$

By appropriately restricting the liberal state, indeterminacy, and falsity components, the Turiyam Neutrosophic Graph reduces to Neutrosophic Graphs and further to Fuzzy Graphs. Thus, the cycle connectivity in a Turiyam Neutrosophic Graph generalizes both the cycle connectivity in Neutrosophic Graphs and the cycle connectivity of Fuzzy Graphs. $\square$

Definition 72 (Cycle Connectivity in Plithogenic Graph). Let $PG = (PM, PN)$ be a plithogenic graph, where:

- Each vertex $v \in PM$ has a membership vector $adf(v) = (a_1(v), a_2(v), \dots, a_s(v)).$
- Each edge $e \in PN$ has a membership vector $bdf(e) = (b_1(e), b_2(e), \dots, b_s(e)).$

For any two vertices $u, v \in PM$, define a set $\theta(u, v)$, called the *cycle evaluation* of u and v , as follows:

$$\theta(u, v) = \{\alpha_i : \alpha_i \in (0, 1], \text{ where}$$

α_i represents the strength of a strong cycle passing through both u and v for each component $i\}$.

If there is no strong cycle containing both u and v , then $\theta(u, v) = \emptyset$.

The *cycle connectivity* between u and v in PG , denoted by $CG_{u,v}$, is defined as:

$$CG_{u,v} = (\max\{\alpha_i : \alpha_i \in \theta_i(u, v)\})_{i=1}^s.$$

If $\theta(u, v) = \emptyset$, then $CG_{u,v} = (0, 0, \dots, 0)$.

The *cycle connectivity* of the plithogenic graph PG , denoted by $CC(PG)$, is defined as:

$$CC(PG) = \max\{CG_{u,v} : u, v \in PM\}.$$

Example 73 (Cycle Connectivity in a Triangle Network with Two Attributes). *Scenario.* Consider a plithogenic triangle graph with vertices A, B, C and edges AB, BC, CA . Let there be $s = 2$ attributes: (1) *capacity*, (2) *security*.

$$\begin{aligned} adf(A) &= (0.9, 0.7), & adf(B) &= (0.8, 0.6), & adf(C) &= (0.7, 0.5), \\ bdf(AB) &= (0.6, 0.5), & bdf(BC) &= (0.5, 0.4), & bdf(CA) &= (0.7, 0.6). \end{aligned}$$

Cycle evaluation. Vertices A and B both belong to the cycle $A-B-C-A$. For component 1 (capacity), the limiting strength of this cycle is the minimum edge membership:

$$\alpha_1 = \min\{0.6, 0.5, 0.7\} = 0.5.$$

For component 2 (security), the limiting strength is

$$\alpha_2 = \min\{0.5, 0.4, 0.6\} = 0.4.$$

Thus,

$$CG_{A,B} = (0.5, 0.4).$$

Similarly, $CG_{B,C} = (0.5, 0.4)$ and $CG_{A,C} = (0.5, 0.4)$ because all pairs share the same cycle.

Graph cycle connectivity.

$$CC(PG) = \max\{CG_{u,v} : u, v \in \{A, B, C\}\} = (0.5, 0.4).$$

The cycle connectivity reveals that the strongest guaranteed joint support for any two vertices is bounded by the weakest edge in the cycle, across attributes. For this network, the cycle connectivity is moderate, suggesting that while redundancy exists (the cycle), the weakest edge limits the system's resilience in both capacity and security.

Theorem 74. *The Plithogenic Cycle Connectivity generalizes the Neutrosophic Cycle Connectivity, Turiyam Neutrosophic Cycle Connectivity, and Fuzzy Cycle Connectivity. Specifically, by choosing appropriate dimensions and components of the membership vectors in the Plithogenic Graph, the Plithogenic Cycle Connectivity reduces to these specific cycle connectivities.*

Proof: Consider a Plithogenic Graph $PG = (PM, PN)$ with vertex membership vectors

$$adf(v) = (a_1(v), a_2(v), \dots, a_s(v))$$

and edge membership vectors

$$bdf(e) = (b_1(e), b_2(e), \dots, b_s(e))$$

.

To recover specific Cycle Connectivities:

(1) *Fuzzy Cycle Connectivity:*

Set $s = 1$, and let $a_1(v) = \sigma(v)$ and $b_1(e) = \mu(e)$, where $\sigma(v)$ and $\mu(e)$ are the vertex and edge membership functions in the fuzzy graph. Then, the definitions of cycle evaluation, cycle connectivity between two vertices, and the cycle connectivity of the graph in the plithogenic context reduce to the definitions in the fuzzy context.

(2) *Neutrosophic Cycle Connectivity:*

Set $s = 3$, and let

$$(a_1(v), a_2(v), a_3(v)) = (\sigma_T(v), \sigma_I(v), \sigma_F(v))$$

, and

$$(b_1(e), b_2(e), b_3(e)) = (\mu_T(e), \mu_I(e), \mu_F(e))$$

. Then, the definitions of cycle evaluation, cycle connectivity between two vertices, and the cycle connectivity of the graph in the plithogenic context reduce to the definitions in the neutrosophic context.

(3) *Turiyam Neutrosophic Cycle Connectivity:*

Set $s = 4$, and let $(a_1(v), a_2(v), a_3(v), a_4(v)) = (t(v), iv(v), fv(v), lv(v))$, and similarly for edge memberships. Then, the definitions reduce accordingly.

Thus, by appropriate choices, the Plithogenic Cycle Connectivity reduces to the specific Cycle Connectivities. $\square$

Theorem 75. *For any Plithogenic Graph $PG = (PM, PN)$, the Cycle Connectivity between any two vertices $u, v \in PM$ is well-defined.*

Proof: In a Plithogenic Graph, the cycles in the underlying crisp graph are well-defined. The strength of a cycle in each component i can be computed based on the membership vectors $a_i(v)$ for vertices and $b_i(e)$ for edges. Therefore, for any two vertices $u, v \in PM$, the set $\theta(u, v)$ is well-defined, and hence the Cycle Connectivity $CG_{u,v}$ is well-defined. $\square$

Theorem 76. *For any two vertices $u, v \in PM$, the components of the Cycle Connectivity $CG_{u,v} = (\alpha_1, \alpha_2, \dots, \alpha_s)$ satisfy*

$$0 \leq \alpha_i \leq 1, \quad \text{for all } i = 1, 2, \dots, s.$$

Proof: The membership values $a_i(v)$ and $b_i(e)$ are in the interval $[0, 1]$. The strength α_i of a cycle is computed using these membership values, typically involving products or minimums, which result in values within $[0, 1]$. Therefore, each α_i satisfies $0 \leq \alpha_i \leq 1$. $\square$

Theorem 77. *Deleting edges from a Plithogenic Graph cannot increase the Cycle Connectivity between any two vertices.*

Proof: Deleting edges may eliminate some cycles that pass through u and v , reducing the set $\theta(u, v)$. Consequently, the maximum strengths α_i in each component cannot increase. Therefore, $CG_{u,v}$ either decreases or remains the same after edge deletion. $\square$

Theorem 78. *Increasing the membership values of edges and vertices in a Plithogenic Graph cannot decrease the Cycle Connectivity between any two vertices.*

Proof: Increasing the membership values $a_i(v)$ and $b_i(e)$ can increase the strengths α_i of cycles passing through u and v . Since the strengths of cycles are functions of these membership values, higher membership values can lead to higher cycle strengths. Therefore, $CG_{u,v}$ cannot decrease when membership values increase. $\square$

3.5 | Hyper-Zagreb Index

The Hyper-Zagreb Index is a topological index of a graph, calculated as the sum of the squares of the degree sums of adjacent vertices [35]. Related concepts include the Zagreb Index [36] and the Multiplicative Zagreb Index [37]. The Fuzzy Hyper-Zagreb Index extends the Hyper-Zagreb Index by incorporating fuzzy conditions [38]. This subsection examines the uncertain Hyper-Zagreb Index, with definitions provided below.

Definition 79 (Hyper-Zagreb Index for a Crisp Graph). [35] Let $G = (V, E)$ be a connected graph, where V is the set of vertices and E is the set of edges. The *Hyper-Zagreb Index (HZI)* of G , denoted by $HZI(G)$, is defined as:

$$HZI(G) = \sum_{uv \in E} [\deg(u) + \deg(v)]^2,$$

where $\deg(u)$ and $\deg(v)$ are the degrees of the vertices u and v , respectively. The index sums the square of the sum of degrees of the endpoints for each edge in the graph.

Definition 80 (Fuzzy Hyper-Zagreb Index (FHZI)). Let $F = (V, E, \sigma, \mu)$ be a connected fuzzy graph where $\sigma : V \rightarrow [0, 1]$ assigns a membership to each vertex and $\mu : E \rightarrow [0, 1]$ assigns a membership to each edge. Define the (fuzzy) degree

$$\deg(u) := \sum_{w: uw \in E} \mu(uw), \quad u \in V.$$

The *Fuzzy Hyper-Zagreb Index* of F is

$$FHZI(F) = \sum_{uv \in E} [\sigma(u) \deg(u) + \sigma(v) \deg(v)]^2.$$

When all vertex memberships are 1 and edges are crisp ($\mu \in \{0, 1\}$), $FHZI$ reduces to the classical Hyper-Zagreb index $\sum_{uv \in E} (d(u) + d(v))^2$.

Example 81 (Fuzzy: Urban Road Reliability Network). *Scenario.* Consider three intersections A – B – C connected in a line. Vertex membership $\sigma(\cdot)$ models the *availability* of an intersection (e.g. signal uptime), and edge membership $\mu(\cdot)$ models the *usable quality* of each road segment.

$$\begin{aligned} V = \{A, B, C\}, \quad E = \{AB, BC\}, \quad \sigma(A) = 0.80, \sigma(B) = 1.00, \sigma(C) = 0.60, \\ \mu(AB) = 0.70, \quad \mu(BC) = 0.50. \end{aligned}$$

Fuzzy degrees.

$$\deg(A) = \mu(AB) = 0.70, \quad \deg(B) = \mu(AB) + \mu(BC) = 0.70 + 0.50 = 1.20, \quad \deg(C) = \mu(BC) = 0.50.$$

FHZI contributions (edge-wise).

$$\begin{aligned} AB : [\sigma(A) \deg(A) + \sigma(B) \deg(B)]^2 &= [0.80 \cdot 0.70 + 1.00 \cdot 1.20]^2 \\ &= [0.56 + 1.20]^2 = (1.76)^2, \\ (1.76)^2 &= \frac{176^2}{100^2} = \frac{30976}{10000} = 3.0976. \\ BC : [\sigma(B) \deg(B) + \sigma(C) \deg(C)]^2 &= [1.00 \cdot 1.20 + 0.60 \cdot 0.50]^2 \\ &= [1.20 + 0.30]^2 = (1.50)^2 = 2.25. \end{aligned}$$

Total.

$$FHZI(F) = 3.0976 + 2.25 = 5.3476.$$

Intersection B has full availability and connects two moderately reliable links, so pairs involving B dominate the index. A larger $FHZI$ indicates corridors where reliable, central hubs concentrate flow—useful for prioritizing maintenance.

Theorem 82. The Hyper-Zagreb Index for a Fuzzy Graph generalizes the Hyper-Zagreb Index for a Crisp Graph.

Proof: To demonstrate that the Hyper-Zagreb Index for a Fuzzy Graph (FHZI) generalizes the Hyper-Zagreb Index for a Crisp Graph (HZI), we show that the latter is a special case of the former when the membership degrees of all vertices and edges in the fuzzy graph are restricted to binary values (0 or 1).

Let $G_c = (V, E)$ be a crisp graph and $G_f = (V, E, \sigma, \mu)$ be a fuzzy graph. Assume that for the fuzzy graph G_f :

$$\sigma(u) = 1, \quad \mu(uv) = 1, \quad \forall u \in V, uv \in E,$$

indicating that all vertices and edges have full membership in the graph.

Under these conditions:

- The degree of a vertex u in the fuzzy graph reduces to its degree in the crisp graph:

$$\deg(u) = \sum_{uv \in E} \mu(uv) = \sum_{uv \in E} 1 = \deg_c(u),$$

where $\deg_c(u)$ denotes the degree of u in the crisp graph G_c .

- The vertex membership $\sigma(u) = 1$ has no scaling effect on the degree of u , so:

$$\sigma(u) \cdot \deg(u) = \deg_c(u).$$

- The FHZI formula for the fuzzy graph becomes:

$$FHZI(G_f) = \sum_{uv \in E} [\sigma(u) \cdot \deg(u) + \sigma(v) \cdot \deg(v)]^2.$$

Substituting $\sigma(u) = 1$, $\deg(u) = \deg_c(u)$, and $\sigma(v) = 1$, $\deg(v) = \deg_c(v)$:

$$FHZI(G_f) = \sum_{uv \in E} [\deg_c(u) + \deg_c(v)]^2.$$

This is precisely the definition of the Hyper-Zagreb Index for a Crisp Graph:

$$HZI(G_c) = \sum_{uv \in E} [\deg_c(u) + \deg_c(v)]^2.$$

By restricting the vertex and edge membership degrees in a fuzzy graph to binary values (0 or 1), the FHZI reduces to the HZI. Therefore, the Hyper-Zagreb Index for a Fuzzy Graph generalizes the Hyper-Zagreb Index for a Crisp Graph. $\square$

Definition 83 (Hyper-Zagreb Index in Neutrosophic Graph). Let $G = (V, E, \sigma, \mu)$ be a connected neutrosophic graph, where:

- Each vertex $v \in V$ is associated with a triple $\sigma(v) = (\sigma_T(v), \sigma_I(v), \sigma_F(v))$, representing truth, indeterminacy, and falsity membership degrees.
- Each edge $e = (u, v) \in E$ is associated with a triple $\mu(e) = (\mu_T(e), \mu_I(e), \mu_F(e))$, representing truth, indeterminacy, and falsity degrees for the connection.

The *Neutrosophic Hyper-Zagreb Index* (NHZI) of the neutrosophic graph G , denoted by $NHZI(G)$, is defined as:

$$NHZI(G) = \sum_{uv \in E} [\sigma_T(u) \cdot \deg_T(u) + \sigma_T(v) \cdot \deg_T(v)]^2,$$

where:

- $\sigma_T(u)$ and $\sigma_T(v)$ are the truth-membership degrees of vertices u and v .
- $\deg_T(u)$ and $\deg_T(v)$ are the truth-based degrees of vertices u and v , calculated as the sum of truth-membership values of edges incident to each vertex.

The NHZI captures the truth-based structural information of the neutrosophic graph.

Example 84 (Neutrosophic: Hospital Supply Links with Data Ambiguity). *Scenario.* Vertices A, B, C are hospitals. For each vertex v , $\sigma_T(v)$ encodes *confirmed operational readiness* (e.g. staffed ICU beds). Edges use μ_T for confirmed route reliability. (Indeterminacy/falsity components may exist but NHZI—as defined—focuses on truth.)

$$V = \{A, B, C\}, \quad E = \{AB, BC\}, \quad \sigma_T(A) = 0.90, \quad \sigma_T(B) = 0.70, \quad \sigma_T(C) = 0.50, \\ \mu_T(AB) = 0.60, \quad \mu_T(BC) = 0.40.$$

Truth-based degrees.

$$\deg_T(A) = 0.60, \quad \deg_T(B) = 0.60 + 0.40 = 1.00, \quad \deg_T(C) = 0.40.$$

NHZI contributions (edge-wise).

$$\begin{aligned} AB : [\sigma_T(A) \deg_T(A) + \sigma_T(B) \deg_T(B)]^2 &= [0.90 \cdot 0.60 + 0.70 \cdot 1.00]^2 \\ &= [0.54 + 0.70]^2 = (1.24)^2, \\ (1.24)^2 &= \frac{124^2}{100^2} = \frac{15376}{10000} = 1.5376. \\ BC : [\sigma_T(B) \deg_T(B) + \sigma_T(C) \deg_T(C)]^2 &= [0.70 \cdot 1.00 + 0.50 \cdot 0.40]^2 \\ &= [0.70 + 0.20]^2 = (0.90)^2 = 0.81. \end{aligned}$$

Total.

$$NHZI(G) = 1.5376 + 0.81 = 2.3476.$$

Hospital B concentrates confirmed capacity and reliable links, so AB contributes most. Higher $NHZI$ flags clusters where assured capacity compounds through dependable routes, supporting robust routing of critical supplies during disruptions.

Theorem 85. *The Hyper-Zagreb Index in a Neutrosophic Graph generalizes the Hyper-Zagreb Index for a Fuzzy Graph and the Hyper-Zagreb Index for a Crisp Graph.*

Proof: To prove that the Hyper-Zagreb Index in a Neutrosophic Graph (NHZI) generalizes both the Hyper-Zagreb Index for a Fuzzy Graph (FHZI) and the Hyper-Zagreb Index for a Crisp Graph (HZI), we will demonstrate that the NHZI reduces to the FHZI and HZI under specific conditions.

Reduction to the Hyper-Zagreb Index for a Fuzzy Graph Let $G_f = (V, E, \sigma, \mu)$ be a fuzzy graph, where $\sigma : V \rightarrow [0, 1]$ is the vertex membership function, and $\mu : E \rightarrow [0, 1]$ is the edge membership function. Consider a neutrosophic graph $G_n = (V, E, \sigma_n, \mu_n)$, where:

$$\sigma_n(v) = (\sigma_T(v), 0, 1 - \sigma_T(v)), \quad \mu_n(e) = (\mu_T(e), 0, 1 - \mu_T(e)).$$

This setup ensures that the neutrosophic graph G_n encodes the fuzzy graph G_f using the truth-membership degrees $\sigma_T(v)$ and $\mu_T(e)$.

In this case:

$$\deg_T(u) = \sum_{uv \in E} \mu_T(uv) = \deg(u),$$

where $\deg(u)$ is the fuzzy degree of vertex u in G_f . Substituting these into the NHZI formula:

$$NHZI(G_n) = \sum_{uv \in E} [\sigma_T(u) \cdot \deg_T(u) + \sigma_T(v) \cdot \deg_T(v)]^2.$$

By substituting $\sigma_T(u) = \sigma(u)$ and $\deg_T(u) = \deg(u)$, we recover the FHZI:

$$FHZI(G_f) = \sum_{uv \in E} [\sigma(u) \cdot \deg(u) + \sigma(v) \cdot \deg(v)]^2.$$

Reduction to the Hyper-Zagreb Index for a Crisp Graph Let $G_c = (V, E)$ be a crisp graph. Consider a neutrosophic graph $G_n = (V, E, \sigma_n, \mu_n)$, where:

$$\sigma_n(v) = (1, 0, 0), \quad \mu_n(e) = (1, 0, 0), \quad \forall v \in V, e \in E.$$

This setup represents the crisp graph G_c in neutrosophic form, with all truth-membership values set to 1.

In this case:

$$\deg_T(u) = \sum_{uv \in E} \mu_T(uv) = \sum_{uv \in E} 1 = \deg(u),$$

where $\deg(u)$ is the degree of vertex u in the crisp graph G_c . Substituting these into the NHZI formula:

$$NHZI(G_n) = \sum_{uv \in E} [\sigma_T(u) \cdot \deg_T(u) + \sigma_T(v) \cdot \deg_T(v)]^2.$$

With $\sigma_T(u) = 1$ and $\deg_T(u) = \deg(u)$, this reduces to the HZI:

$$HZI(G_c) = \sum_{uv \in E} [\deg(u) + \deg(v)]^2.$$

By varying the membership values in a neutrosophic graph, the NHZI can represent both the FHZI and HZI as special cases. Therefore, the Hyper-Zagreb Index in a Neutrosophic Graph generalizes the Hyper-Zagreb Index for a Fuzzy Graph and the Hyper-Zagreb Index for a Crisp Graph. $\square$

Definition 86 (Hyper-Zagreb Index in Turiyam Neutrosophic Graph). Let $G = (V, E, \sigma, \mu)$ be a connected Turiyam Neutrosophic graph, where:

- Each vertex $v \in V$ is associated with a quadruple $\sigma(v) = (t(v), iv(v), fv(v), lv(v))$, representing truth, indeterminacy, falsity, and liberal state values.
- Each edge $e = (u, v) \in E$ is associated with a quadruple $\mu(e) = (t(e), iv(e), fv(e), lv(e))$, representing truth, indeterminacy, falsity, and liberal state values for the connection.

The *Turiyam Neutrosophic Hyper-Zagreb Index (THZI)* of the Turiyam Neutrosophic graph G , denoted by $THZI(G)$, is defined as:

$$THZI(G) = \sum_{uv \in E} [t(u) \cdot \deg_T(u) + t(v) \cdot \deg_T(v)]^2,$$

where:

- $t(u)$ and $t(v)$ are the truth values of vertices u and v .
- $\deg_T(u)$ and $\deg_T(v)$ are the truth-based degrees of vertices u and v , calculated as the sum of truth values of edges incident to each vertex.

The THZI focuses on the truth-based structural information of the Turiyam Neutrosophic graph.

Theorem 87. *The Hyper-Zagreb Index in a Turiyam Neutrosophic Graph generalizes the Hyper-Zagreb Index in a Neutrosophic Graph and the Hyper-Zagreb Index in a Fuzzy Graph.*

Proof: To prove this theorem, we demonstrate that the Turiyam Neutrosophic Hyper-Zagreb Index (THZI) reduces to the Neutrosophic Hyper-Zagreb Index (NHZI) and the Fuzzy Hyper-Zagreb Index (FHZI) under specific conditions.

Reduction to the Neutrosophic Hyper-Zagreb Index Let $G_n = (V, E, \sigma_n, \mu_n)$ be a neutrosophic graph, where:

$$\sigma_n(v) = (t(v), iv(v), fv(v)), \quad \mu_n(e) = (t(e), iv(e), fv(e)).$$

Consider a Turiyam Neutrosophic graph $G_t = (V, E, \sigma_t, \mu_t)$, where:

$$\sigma_t(v) = (t(v), iv(v), fv(v), 0), \quad \mu_t(e) = (t(e), iv(e), fv(e), 0).$$

Here, the liberal state value $lv(v)$ and $lv(e)$ are set to 0, effectively reducing G_t to G_n .

In this case, the truth-based degree $\deg_T(u)$ remains the same for both graphs:

$$\deg_T(u) = \sum_{uv \in E} t(uv).$$

Substituting these values into the THZI definition:

$$THZI(G_t) = \sum_{uv \in E} [t(u) \cdot \deg_T(u) + t(v) \cdot \deg_T(v)]^2.$$

This matches the definition of the NHZI for G_n , showing that the THZI reduces to the NHZI when $lv(v) = 0$.
Reduction to the Fuzzy Hyper-Zagreb Index Let $G_f = (V, E, \sigma_f, \mu_f)$ be a fuzzy graph, where:

$$\sigma_f(v) = \sigma(v), \quad \mu_f(e) = \mu(e).$$

Consider a Turiyam Neutrosophic graph $G_t = (V, E, \sigma_t, \mu_t)$, where:

$$\sigma_t(v) = (\sigma(v), 0, 1 - \sigma(v), 0), \quad \mu_t(e) = (\mu(e), 0, 1 - \mu(e), 0).$$

This setup encodes the fuzzy graph G_f into the truth-membership dimension of G_t .

For the truth-based degree:

$$\deg_T(u) = \sum_{uv \in E} t(uv) = \sum_{uv \in E} \mu(uv).$$

Substituting these values into the THZI definition:

$$THZI(G_t) = \sum_{uv \in E} [t(u) \cdot \deg_T(u) + t(v) \cdot \deg_T(v)]^2.$$

By setting $t(u) = \sigma(u)$, $t(v) = \sigma(v)$, and $t(uv) = \mu(uv)$, this matches the FHZI for G_f :

$$FHZI(G_f) = \sum_{uv \in E} [\sigma(u) \cdot \deg(u) + \sigma(v) \cdot \deg(v)]^2.$$

By varying the membership values in the Turiyam Neutrosophic graph, the THZI can represent both the NHZI and FHZI as special cases. Therefore, the Hyper-Zagreb Index in a Turiyam Neutrosophic Graph generalizes the Hyper-Zagreb Index in a Neutrosophic Graph and the Hyper-Zagreb Index in a Fuzzy Graph. $\square$

Definition 88 (Hyper-Zagreb Index in Plithogenic Graph). Let $PG = (PM, PN)$ be a connected plithogenic graph, where:

- Each vertex $v \in PM$ is associated with a membership vector $adf(v) = (a_1(v), a_2(v), \dots, a_s(v))$, representing multiple attributes.
- Each edge $e \in PN$ is associated with a membership vector $bdf(e) = (b_1(e), b_2(e), \dots, b_s(e))$, representing multiple attributes of the edge.

The *Plithogenic Hyper-Zagreb Index* (PHZI) of the plithogenic graph PG , denoted by $PHZI(PG)$, is defined as:

$$PHZI(PG) = \sum_{uv \in PN} \left[\sum_{i=1}^s (a_i(u) \cdot \deg_i(u) + a_i(v) \cdot \deg_i(v)) \right]^2,$$

where:

- $a_i(u)$ and $a_i(v)$ are the i -th components of the membership vectors of vertices u and v .
- $\deg_i(u)$ and $\deg_i(v)$ are the i -th component-based degrees of vertices u and v , calculated as the sum of the i -th component values of edges incident to each vertex.

The PHZI considers multiple attributes, extending the Hyper-Zagreb Index to the plithogenic environment.

Example 89 (PHZI on a Two-Attribute Path Network (Capacity & Security)). Consider the path $A-B-C$ with $s = 2$ attributes:

$$adf(A) = (0.8, 0.4), \quad adf(B) = (1.0, 0.6), \quad adf(C) = (0.7, 0.5),$$

$$bdf(AB) = (0.6, 0.4), \quad bdf(BC) = (0.5, 0.3),$$

which satisfy typical plithogenic constraints $b_i(uv) \leq \min\{a_i(u), a_i(v)\}$ for each component i .

Component-wise degrees.

$$\deg_1(A) = 0.6, \quad \deg_1(B) = 0.6 + 0.5 = 1.1, \quad \deg_1(C) = 0.5; \quad \deg_2(A) = 0.4, \quad \deg_2(B) = 0.4 + 0.3 = 0.7, \quad \deg_2(C) = 0.3.$$

Edge AB contribution.

$$\begin{aligned} \text{Sum over } i : \underbrace{0.8 \cdot 0.6 + 1.0 \cdot 1.1}_{i=1} &= 0.48 + 1.10 = 1.58, \quad \underbrace{0.4 \cdot 0.4 + 0.6 \cdot 0.7}_{i=2} = 0.16 + 0.42 = 0.58, \\ \Rightarrow S_{AB} &= 1.58 + 0.58 = 2.16, \quad S_{AB}^2 = (2.16)^2 = \left(\frac{216}{100}\right)^2 = \frac{46656}{10000} = 4.6656. \end{aligned}$$

Edge BC contribution.

$$\begin{aligned} \text{Sum over } i : \underbrace{1.0 \cdot 1.1 + 0.7 \cdot 0.5}_{i=1} &= 1.10 + 0.35 = 1.45, \quad \underbrace{0.6 \cdot 0.7 + 0.5 \cdot 0.3}_{i=2} = 0.42 + 0.15 = 0.57, \\ \Rightarrow S_{BC} &= 1.45 + 0.57 = 2.02, \quad S_{BC}^2 = (2.02)^2 = \left(\frac{202}{100}\right)^2 = \frac{40804}{10000} = 4.0804. \end{aligned}$$

Total $PHZI$.

$$PHZI(PG) = 4.6656 + 4.0804 = 8.7460.$$

Interpretation. Vertex B is central and strongly weighted in both attributes, and it connects two reasonably strong edges; hence both edge terms are large, reflecting a corridor where multi-attribute importance compounds along the path.

Theorem 90. *The Plithogenic Hyper-Zagreb Index generalizes the Neutrosophic Hyper-Zagreb Index, Turiyam Neutrosophic Hyper-Zagreb Index, and Fuzzy Hyper-Zagreb Index. Specifically, by choosing appropriate dimensions and components of the membership vectors, the $PHZI$ reduces to these specific indices.*

Proof: Consider a Plithogenic Graph $PG = (PM, PN)$ with membership vectors

$$adf(v) = (a_1(v), a_2(v), \dots, a_s(v))$$

for vertices and

$$bdf(e) = (b_1(e), b_2(e), \dots, b_s(e))$$

for edges.

To recover specific Hyper-Zagreb Indices:

(1) *Fuzzy Hyper-Zagreb Index (FHZI):*

Set $s = 1$, and let $a_1(v) = \sigma(v)$ and $b_1(e) = \mu(e)$, where $\sigma(v)$ and $\mu(e)$ are the vertex and edge membership functions in the fuzzy graph.

The degree of a vertex u becomes:

$$\deg_1(u) = \sum_{e=(u,w) \in PN} b_1(e) = \sum_{e=(u,w) \in PN} \mu(e).$$

The $PHZI$ reduces to:

$$PHZI(PG) = \sum_{uv \in PN} [\sigma(u) \cdot \deg(u) + \sigma(v) \cdot \deg(v)]^2,$$

which matches the definition of the Fuzzy Hyper-Zagreb Index.

(2) *Neutrosophic Hyper-Zagreb Index (NHZI):*

Set $s = 3$, and let:

$$\begin{aligned} (a_1(v), a_2(v), a_3(v)) &= (\sigma_T(v), \sigma_I(v), \sigma_F(v)), \\ (b_1(e), b_2(e), b_3(e)) &= (\mu_T(e), \mu_I(e), \mu_F(e)). \end{aligned}$$

Focusing on the truth component ($i = 1$), the degree becomes:

$$\deg_1(u) = \sum_{e=(u,w) \in PN} \mu_T(e).$$

The PHZI reduces to:

$$PHZI(PG) = \sum_{uv \in PN} [\sigma_T(u) \cdot \deg_T(u) + \sigma_T(v) \cdot \deg_T(v)]^2,$$

which corresponds to the Neutrosophic Hyper-Zagreb Index.

(3) *Turiyam Neutrosophic Hyper-Zagreb Index (THZI)*:

Set $s = 4$, and let:

$$\begin{aligned} (a_1(v), a_2(v), a_3(v), a_4(v)) &= (t(v), iv(v), fv(v), lv(v)), \\ (b_1(e), b_2(e), b_3(e), b_4(e)) &= (t(e), iv(e), fv(e), lv(e)). \end{aligned}$$

Focusing on the truth component ($i = 1$), the degree becomes:

$$\deg_1(u) = \sum_{e=(u,w) \in PN} t(e).$$

The PHZI reduces to:

$$PHZI(PG) = \sum_{uv \in PN} [t(u) \cdot \deg_T(u) + t(v) \cdot \deg_T(v)]^2,$$

matching the Turiyam Neutrosophic Hyper-Zagreb Index.

Thus, by choosing appropriate dimensions and components, the PHZI reduces to the specific indices. $\square$

Theorem 91. *The Plithogenic Hyper-Zagreb Index $PHZI(PG)$ is always non-negative, i.e.,*

$$PHZI(PG) \geq 0.$$

Proof: The PHZI is defined as a sum of squared terms:

$$PHZI(PG) = \sum_{uv \in PN} (S_{uv})^2,$$

where

$$S_{uv} = \sum_{i=1}^s (a_i(u) \cdot \deg_i(u) + a_i(v) \cdot \deg_i(v)).$$

Since each $(S_{uv})^2 \geq 0$, the sum of non-negative terms is non-negative. Therefore, $PHZI(PG) \geq 0$. $\square$

Theorem 92. *Increasing the membership values $a_i(v)$ of a vertex v increases the Plithogenic Hyper-Zagreb Index $PHZI(PG)$.*

Proof: Consider the effect of increasing $a_i(v)$ on $PHZI(PG)$. The term $a_i(v) \cdot \deg_i(v)$ appears in S_{uv} for all edges uv incident to v . Increasing $a_i(v)$ increases S_{uv} , thus increasing $(S_{uv})^2$. Since $PHZI(PG)$ is the sum of these squared terms, it increases when $a_i(v)$ increases. $\square$

Theorem 93. *Increasing the membership values $b_i(e)$ of an edge e increases the Plithogenic Hyper-Zagreb Index $PHZI(PG)$.*

Proof: Increasing $b_i(e)$ affects the degrees $\deg_i(u)$ and $\deg_i(v)$ of the vertices u and v incident to e . This, in turn, increases the terms $a_i(u) \cdot \deg_i(u)$ and $a_i(v) \cdot \deg_i(v)$ in S_{uv} . As a result, $(S_{uv})^2$ increases, leading to an increase in $PHZI(PG)$. $\square$

Theorem 94. *The Plithogenic Hyper-Zagreb Index $PHZI(PG)$ satisfies the following upper bound:*

$$PHZI(PG) \leq \left(2 \cdot \max_{v \in PM} \left\{ \sum_{i=1}^s a_i(v) \cdot \deg_i(v) \right\} \right)^2 \cdot |PN|,$$

where $|PN|$ is the number of edges in the plithogenic graph.

Proof: For each edge $uv \in PN$, the term S_{uv} satisfies:

$$S_{uv} = \sum_{i=1}^s (a_i(u) \cdot \deg_i(u) + a_i(v) \cdot \deg_i(v)) \leq 2 \cdot \max_{v \in PM} \left\{ \sum_{i=1}^s a_i(v) \cdot \deg_i(v) \right\}.$$

Therefore,

$$(S_{uv})^2 \leq \left(2 \cdot \max_{v \in PM} \left\{ \sum_{i=1}^s a_i(v) \cdot \deg_i(v) \right\} \right)^2.$$

Summing over all edges $uv \in PN$, we get:

$$PHZI(PG) \leq \left(2 \cdot \max_{v \in PM} \left\{ \sum_{i=1}^s a_i(v) \cdot \deg_i(v) \right\} \right)^2 \cdot |PN|.$$

□

Theorem 95. Adding an edge $e = (u, v)$ with membership vector $bdf(e)$ to the plithogenic graph PG increases the Plithogenic Hyper-Zagreb Index $PHZI(PG)$.

Proof: Adding an edge $e = (u, v)$ introduces a new term $(S_{uv})^2$ in $PHZI(PG)$, where S_{uv} is calculated using the updated degrees $\deg_i(u)$ and $\deg_i(v)$. The degrees $\deg_i(u)$ and $\deg_i(v)$ increase due to the new edge, increasing the terms $a_i(u) \cdot \deg_i(u)$ and $a_i(v) \cdot \deg_i(v)$. Therefore, $PHZI(PG)$ increases with the addition of the new edge. □

3.6 | Linear Forest

This subsection provides an explanation of the concept of a Linear Forest. A linear forest is defined as a disjoint union of paths, comprising one or more disconnected paths without any cycles [39, 40]. The formal definitions, along with related concepts, are presented below.

Definition 96 (Forest). [41] A *forest* is an undirected graph that contains no cycles. Equivalently, it is a disjoint union of one or more trees. Formally, a graph $G = (V, E)$ is called a forest if it is acyclic and can be represented as a disjoint union of trees:

$$G = T_1 + T_2 + \cdots + T_k,$$

where T_i are trees and $+$ denotes the disjoint union of graphs. In other words, a forest is a collection of one or more disconnected trees.

Definition 97 (Linear Forest). [40] A *linear forest* is a graph that is the disjoint union of a collection of paths. Formally, a graph $G = (V, E)$ is called a linear forest if there exists a collection of paths $\{P_1, P_2, \dots, P_k\}$ such that:

$$G = P_1 + P_2 + \cdots + P_k,$$

where $+$ denotes the disjoint union of graphs. In other words, a linear forest consists of one or more disconnected paths.

Theorem 98. A Linear Forest is a Forest.

Proof: The statement holds trivially. □

Definition 99 (Fuzzy Path FP_n). A *fuzzy path* FP_n is a fuzzy graph in which the underlying crisp graph is a path P_n , and the vertex and edge membership values satisfy:

- $\sigma(v) > 0$ for all v in the path,
- $\mu(u, v) > 0$ for all edges (u, v) in the path.

Definition 100 (Fuzzy Linear Forest). A *fuzzy linear forest* is a fuzzy graph that is the disjoint union of fuzzy paths. Formally, a fuzzy graph $G = (V, E, \sigma, \mu)$ is called a fuzzy linear forest if there exists a collection of fuzzy paths $\{FP_1, FP_2, \dots, FP_k\}$ such that:

$$G = FP_1 + FP_2 + \dots + FP_k,$$

where $+$ denotes the disjoint union of fuzzy graphs. In other words, a fuzzy linear forest consists of one or more disconnected fuzzy paths.

Theorem 101. *A Fuzzy Linear Forest generalizes the concept of a Linear Forest.*

Proof: In a crisp graph G^* , a Linear Forest is defined as the disjoint union of paths $\{P_1, P_2, \dots, P_k\}$, where each P_i is a simple path, and $G^* = P_1 + P_2 + \dots + P_k$. Each path consists of vertices and edges, with no cycles.

In a fuzzy graph $G = (V, E, \sigma, \mu)$, a Fuzzy Linear Forest is defined as the disjoint union of fuzzy paths $\{FP_1, FP_2, \dots, FP_k\}$, where each fuzzy path FP_i satisfies the following:

- $\sigma(v) > 0$ for all vertices v in FP_i , indicating the presence of these vertices in the fuzzy path.
- $\mu(e) > 0$ for all edges e in FP_i , indicating the presence of these edges in the fuzzy path.
- There are no cycles in FP_i , ensuring that it is a valid fuzzy path.

When $\sigma(v) \in \{0, 1\}$ and $\mu(e) \in \{0, 1\}$, the fuzzy graph G reduces to the crisp graph G^* . Under these constraints, a Fuzzy Linear Forest becomes a Linear Forest, as all membership values are binary, and the structure aligns with the definition of a Linear Forest.

The Fuzzy Linear Forest allows for partial membership of vertices and edges ($\sigma(v), \mu(e) \in [0, 1]$), which extends the concept of Linear Forests by incorporating uncertainty and degrees of membership. Thus, every Linear Forest is a special case of a Fuzzy Linear Forest when the membership degrees are binary.

Since a Fuzzy Linear Forest reduces to a Linear Forest under binary membership values, it generalizes the concept of a Linear Forest. $\square$

Definition 102 (Neutrosophic Linear Forest). A *Neutrosophic Linear Forest* is a neutrosophic graph that is the disjoint union of neutrosophic paths. Formally, let $G = (V, E, \sigma, \mu)$ be a neutrosophic graph, where:

- Each vertex $v \in V$ is assigned a triple $\sigma(v) = (\sigma_T(v), \sigma_I(v), \sigma_F(v))$, representing truth, indeterminacy, and falsity membership degrees.
- Each edge $e = (u, v) \in E$ is assigned a triple $\mu(e) = (\mu_T(e), \mu_I(e), \mu_F(e))$, representing truth, indeterminacy, and falsity degrees.

G is called a neutrosophic linear forest if there exists a collection of disjoint neutrosophic paths

$$\{NP_1, NP_2, \dots, NP_k\}$$

such that:

$$G = NP_1 + NP_2 + \dots + NP_k,$$

where $+$ denotes the disjoint union of neutrosophic graphs. A neutrosophic path NP_n is a neutrosophic graph where the underlying crisp graph is a path and all vertices and edges have non-zero truth, indeterminacy, and falsity membership values.

Theorem 103. *A Neutrosophic Linear Forest generalizes the concepts of a Fuzzy Linear Forest and a Linear Forest.*

Proof: To show that a Neutrosophic Linear Forest generalizes both a Fuzzy Linear Forest and a Linear Forest, we demonstrate that each of the latter two can be represented as a special case of the former under specific constraints on membership values.

Step 1: Reduction to Linear Forest Consider a neutrosophic graph $G = (V, E, \sigma, \mu)$, where:

$$\sigma(v) = (\sigma_T(v), \sigma_I(v), \sigma_F(v)), \quad \mu(e) = (\mu_T(e), \mu_I(e), \mu_F(e)).$$

If the membership functions are restricted such that:

$$\sigma_T(v) \in \{0, 1\}, \quad \sigma_I(v) = 0, \quad \sigma_F(v) = 1 - \sigma_T(v),$$

and

$$\mu_T(e) \in \{0, 1\}, \quad \mu_I(e) = 0, \quad \mu_F(e) = 1 - \mu_T(e),$$

then the neutrosophic graph G reduces to a crisp graph $G^* = (V, E)$. Under this constraint:

- $\sigma_T(v) = 1$ indicates the presence of vertex v .
- $\mu_T(e) = 1$ indicates the presence of edge e .

If G is a Neutrosophic Linear Forest under these constraints, it is equivalent to a Linear Forest, as the underlying crisp graph consists of disjoint paths.

Step 2: Reduction to Fuzzy Linear Forest Now consider the case where the indeterminacy and falsity components of the neutrosophic graph are zero:

$$\sigma_I(v) = 0, \quad \sigma_F(v) = 0, \quad \mu_I(e) = 0, \quad \mu_F(e) = 0.$$

In this case, the neutrosophic graph G reduces to a fuzzy graph $G_F = (V, E, \sigma_T, \mu_T)$, where:

$$\sigma_T(v) \in [0, 1], \quad \mu_T(e) \in [0, 1].$$

If G is a Neutrosophic Linear Forest under these constraints, it is equivalent to a Fuzzy Linear Forest, as the graph consists of disjoint fuzzy paths with partial membership values.

Step 3: Generalization by Neutrosophic Linear Forest The Neutrosophic Linear Forest extends the concepts of Linear Forest and Fuzzy Linear Forest by allowing vertices and edges to have three membership components: truth (σ_T, μ_T), indeterminacy (σ_I, μ_I), and falsity (σ_F, μ_F). These components enable the representation of uncertainty and partial presence in a more nuanced way:

- When all vertices and edges have non-zero indeterminacy or falsity values, the structure captures ambiguous or contradictory information, which cannot be represented by a Fuzzy Linear Forest or a Linear Forest.
- The Neutrosophic Linear Forest allows for the disjoint union of neutrosophic paths, each characterized by a richer set of attributes than its fuzzy or crisp counterparts.

The Neutrosophic Linear Forest generalizes the Fuzzy Linear Forest and the Linear Forest, as both can be obtained as special cases under appropriate constraints on the membership functions. $\square$

Definition 104 (Turiyam Neutrosophic Linear Forest). A *Turiyam Neutrosophic Linear Forest* is a Turiyam Neutrosophic graph that is the disjoint union of Turiyam Neutrosophic paths. Formally, let $G = (V, E, \sigma, \mu)$ be a Turiyam Neutrosophic graph, where:

- Each vertex $v \in V$ is assigned a quadruple $\sigma(v) = (t(v), iv(v), fv(v), lv(v))$, representing truth, indeterminacy, falsity, and liberal state values.
- Each edge $e = (u, v) \in E$ is assigned a quadruple $\mu(e) = (t(e), iv(e), fv(e), lv(e))$, representing truth, indeterminacy, falsity, and liberal state degrees.

G is called a Turiyam Neutrosophic linear forest if there exists a collection of disjoint Turiyam Neutrosophic paths $\{TP_1, TP_2, \dots, TP_k\}$ such that:

$$G = TP_1 + TP_2 + \dots + TP_k,$$

where $+$ denotes the disjoint union of Turiyam Neutrosophic graphs. A Turiyam Neutrosophic path TP_n is a Turiyam Neutrosophic graph where the underlying crisp graph is a path and all vertices and edges have non-zero values in truth, indeterminacy, falsity, and liberal state components.

Theorem 105. A *Turiyam Neutrosophic Linear Forest* generalizes the concepts of a *Neutrosophic Linear Forest*, a *Fuzzy Linear Forest*, and a *Linear Forest*.

Proof: To prove that a Turiyam Neutrosophic Linear Forest generalizes the three mentioned concepts, we show that each of them can be obtained as a special case under specific constraints on the membership values of vertices and edges.

Consider a Turiyam Neutrosophic graph $G = (V, E, \sigma, \mu)$, where:

$$\sigma(v) = (t(v), iv(v), fv(v), lv(v)), \quad \mu(e) = (t(e), iv(e), fv(e), lv(e)).$$

If the membership functions are restricted such that:

$$t(v), t(e) \in \{0, 1\}, \quad iv(v) = 0, \quad fv(v) = 1 - t(v), \quad lv(v) = 0,$$

and similarly for edges, the Turiyam Neutrosophic graph G reduces to a crisp graph $G^* = (V, E)$. Under this constraint:

- $t(v) = 1$ indicates the presence of vertex v ,
- $t(e) = 1$ indicates the presence of edge e ,

and the other components contribute no additional information. If G is a Turiyam Neutrosophic Linear Forest under these constraints, it reduces to a Linear Forest, as the underlying crisp graph consists of disjoint paths.

Now consider the case where the indeterminacy (iv), falsity (fv), and liberal state (lv) components are zero:

$$iv(v) = 0, \quad fv(v) = 0, \quad lv(v) = 0, \quad iv(e) = 0, \quad fv(e) = 0, \quad lv(e) = 0.$$

In this scenario, the Turiyam Neutrosophic graph G reduces to a fuzzy graph $G_F = (V, E, t, t)$, where:

$$t(v) \in [0, 1], \quad t(e) \in [0, 1].$$

If G is a Turiyam Neutrosophic Linear Forest under these constraints, it is equivalent to a Fuzzy Linear Forest, as the graph consists of disjoint fuzzy paths with partial membership values.

Next, consider the case where the liberal state component (lv) is zero:

$$lv(v) = 0, \quad lv(e) = 0.$$

In this case, the Turiyam Neutrosophic graph G reduces to a Neutrosophic graph $G_N = (V, E, (t, iv, fv), (t, iv, fv))$, where each vertex and edge has three components: truth (t), indeterminacy (iv), and falsity (fv). If G is a Turiyam Neutrosophic Linear Forest under these constraints, it is equivalent to a Neutrosophic Linear Forest, as the disjoint paths have richer membership representations than fuzzy or crisp graphs.

The Turiyam Neutrosophic Linear Forest extends these concepts by allowing vertices and edges to have four membership components: truth (t), indeterminacy (iv), falsity (fv), and liberal state (lv). This representation enables:

- Enhanced modeling capabilities, capturing a broader range of uncertainties, contradictions, and additional attributes (liberal state).
- Disjoint unions of Turiyam Neutrosophic paths with richer semantic interpretations for real-world applications.

The Turiyam Neutrosophic Linear Forest generalizes the Neutrosophic Linear Forest, the Fuzzy Linear Forest, and the Linear Forest, as each of them can be obtained as a special case under appropriate constraints on the membership functions. $\square$

Definition 106 (Plithogenic Linear Forest). A *Plithogenic Linear Forest* is a plithogenic graph that is the disjoint union of plithogenic paths. Formally, let $PG = (PM, PN)$ be a plithogenic graph, where:

- Each vertex $v \in PM$ is associated with a membership vector $adf(v) = (a_1(v), a_2(v), \dots, a_s(v))$, representing multiple attributes.
- Each edge $e \in PN$ is associated with a membership vector $bdf(e) = (b_1(e), b_2(e), \dots, b_s(e))$, representing multiple attributes of the edge.

PG is called a plithogenic linear forest if there exists a collection of disjoint plithogenic paths $\{PP_1, PP_2, \dots, PP_k\}$ such that:

$$PG = PP_1 + PP_2 + \dots + PP_k,$$

where $+$ denotes the disjoint union of plithogenic graphs. A plithogenic path PP_n is a plithogenic graph where the underlying crisp graph is a path and all vertices and edges have non-zero values in each component of the membership vector.

Example 107 (Two-Component Plithogenic Linear Forest). Let the underlying crisp graph be the disjoint union $U-V-W \sqcup X-Y$. Take $s = 2$ attributes (e.g., *load* and *security*):

$$\begin{aligned} adf(U) &= (0.7, 0.5), & adf(V) &= (0.9, 0.6), & adf(W) &= (0.6, 0.4), & bdf(UV) &= (0.6, 0.5), \\ adf(X) &= (0.8, 0.5), & adf(Y) &= (0.7, 0.7), & & & bdf(VW) &= (0.5, 0.3), \\ & & & & & & bdf(XY) &= (0.6, 0.5). \end{aligned}$$

These satisfy $b_i(uv) \leq \min\{a_i(u), a_i(v)\}$ for each component i . Component-wise degrees illustrate the structure:

$$\deg_1(U, V, W, X, Y) = (0.6, 1.1, 0.5, 0.6, 0.6), \quad \deg_2(U, V, W, X, Y) = (0.5, 0.8, 0.3, 0.5, 0.5).$$

Since each connected component is a path and every vertex has crisp degree ≤ 2 , this PG is a plithogenic linear forest. The component-wise degrees provide the multi-attribute view of how “strength” distributes along each path.

Theorem 108. *A Plithogenic Linear Forest is a Plithogenic Graph.*

Proof: The statement holds trivially. □

Theorem 109. *A Plithogenic Linear Forest is a Plithogenic Forest.*

Proof: The statement holds trivially. □

Theorem 110. *The Plithogenic Linear Forest generalizes the Neutrosophic Linear Forest, Turiyam Neutrosophic Linear Forest, Fuzzy Linear Forest, and Classic Linear Forest. Specifically, by choosing appropriate dimensions and components of the membership vectors, a Plithogenic Linear Forest reduces to these specific linear forests.*

Proof: Consider a Plithogenic Linear Forest $PG = (PM, PN)$ with membership vectors

$$adf(v) = (a_1(v), a_2(v), \dots, a_s(v))$$

for vertices and

$$bdf(e) = (b_1(e), b_2(e), \dots, b_s(e))$$

for edges.

To recover specific Linear Forests:

(1) *Classic Linear Forest:*

Set $s = 0$, meaning there are no membership vectors. The plithogenic graph PG reduces to the classic graph $G = (V, E)$, where $V = PM$ and $E = PN$. The disjoint union of paths in PG corresponds directly to the classic linear forest.

(2) *Fuzzy Linear Forest:*

Set $s = 1$, and let $a_1(v) = \sigma(v)$ and $b_1(e) = \mu(e)$, where $\sigma(v)$ and $\mu(e)$ are the vertex and edge membership functions in the fuzzy graph. The plithogenic paths PP_i correspond to fuzzy paths FP_i , and their disjoint union forms a fuzzy linear forest.

(3) *Neutrosophic Linear Forest:*

Set $s = 3$, and let:

$$(a_1(v), a_2(v), a_3(v)) = (\sigma_T(v), \sigma_I(v), \sigma_F(v)),$$

$$(b_1(e), b_2(e), b_3(e)) = (\mu_T(e), \mu_I(e), \mu_F(e)).$$

The plithogenic paths PP_i correspond to neutrosophic paths NP_i , and their disjoint union forms a neutrosophic linear forest.

(4) *Turiyam Neutrosophic Linear Forest:*

Set $s = 4$, and let:

$$(a_1(v), a_2(v), a_3(v), a_4(v)) = (t(v), iv(v), fv(v), lv(v)),$$

$$(b_1(e), b_2(e), b_3(e), b_4(e)) = (t(e), iv(e), fv(e), lv(e)).$$

The plithogenic paths PP_i correspond to Turiyam Neutrosophic paths TP_i , and their disjoint union forms a Turiyam Neutrosophic linear forest.

Thus, by choosing appropriate dimensions and components, the Plithogenic Linear Forest reduces to the specific linear forests. $\square$

Theorem 111. *For any collection of disjoint plithogenic paths $\{PP_1, PP_2, \dots, PP_k\}$, their disjoint union $PG = PP_1 + PP_2 + \dots + PP_k$ forms a Plithogenic Linear Forest.*

Proof: By definition, a Plithogenic Linear Forest is the disjoint union of plithogenic paths. Since each PP_i is a plithogenic path, and the union is disjoint (no shared vertices or edges), PG is a Plithogenic Linear Forest. $\square$

Theorem 112. *Any subgraph of a Plithogenic Linear Forest is also a Plithogenic Linear Forest or a plithogenic tree.*

Proof: A Plithogenic Linear Forest consists of disjoint plithogenic paths. Any subgraph of a path is either a path or a tree (since removing edges from a path cannot introduce cycles). Therefore, any subgraph of PG is either a plithogenic path, a collection of plithogenic paths, or a plithogenic tree (in case it becomes disconnected). Thus, it is a Plithogenic Linear Forest or a plithogenic tree. $\square$

Definition 113 (Acyclic in Plithogenic Graph). A plithogenic graph G is said to be *acyclic* if its underlying crisp graph $G^* = (V, E)$ contains no cycles, where:

$G^* = (V, E)$ is derived by disregarding the plithogenic structure and considering only the vertices and edges.

Theorem 114. *A Plithogenic Linear Forest is an acyclic plithogenic graph.*

Proof: Since a Plithogenic Linear Forest is composed of disjoint plithogenic paths, and paths are acyclic by definition, PG contains no cycles. Therefore, PG is acyclic. $\square$

Theorem 115. *The connected components of a Plithogenic Linear Forest are plithogenic paths.*

Proof: By definition, a Plithogenic Linear Forest is the disjoint union of plithogenic paths PP_i . Each PP_i is a connected component, and since they are paths, the connected components of PG are plithogenic paths. $\square$

Theorem 116. *Let PG be a Plithogenic Linear Forest with n vertices and k components (plithogenic paths). Then the number of edges m in the underlying crisp graph satisfies:*

$$m = n - k.$$

Proof: In each plithogenic path PP_i with n_i vertices, there are $n_i - 1$ edges. Summing over all k paths:

$$m = \sum_{i=1}^k (n_i - 1) = \left(\sum_{i=1}^k n_i \right) - k = n - k.$$

□

4 | Conclusion and Future Work

In this study we introduced and examined advanced notions—including *Range Labeling*, *Cycle Connectivity*, *Linear Forests*, *Product Graphs*, *Median Graphs*, and the *Hyper-Zagreb Index*—within the settings of *Fuzzy Graphs*, *Neutrosophic Graphs*, and *Turiyam Neutrosophic Graphs*.

Looking forward, we expect fruitful developments in three main directions:

- (1) large-scale computational experiments,
- (2) the design and analysis of dedicated graph algorithms, and
- (3) the exploration of further specialised classes of graphs.

We also hope to see progress on extended models that employ *Hypergraphs* [42, 43] and *SuperHypergraphs* [44, 45].

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