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Toward a Formal Framework of Physical SuperHyperStructures: Multi-Layered Extensions of Hyperstructures in Physics

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Abstract

A *hyperstructure* replaces ordinary single-valued operations by set-valued ones: outcomes of interactions are subsets of a universe, naturally organized through the powerset construction. Pushing this idea further, a *superhyperstructure* works with iterated powersets (the n -fold powerset) so that one can represent genuinely hierarchical collections of possibilities across multiple levels.

In this work we adapt these frameworks to physical modeling and present concrete motivating examples. A *Physical HyperStructure* takes a set of physical states as its base and equips it with interaction hyperoperations subject to physically meaningful constraints, such as conservation (possibly with bounded losses) and covariance under a prescribed symmetry group. A *Physical SuperHyperStructure* then lifts the same principles to iterated powersets, allowing operations on ensembles and higher-order families of ensembles while still enforcing conservation-type bounds and symmetry consistency across levels.

Keywords: HyperStructure, SuperHyperStructure, Physical HyperStructure, Physical SuperHyperStructure

1 | Preliminaries

This section fixes the terminology and notation used in the sequel. Unless we explicitly say otherwise, all objects are considered in the finite setting. In addition, whenever a powerset (or an iterated powerset) is formed, we allow the empty set as an admissible member by convention.

1.1 | Hyperstructure and Superhyperstructure

Hyperstructures provide a convenient language for situations in which combining elements may yield several possible outcomes rather than a single deterministic result. A standard way to formalize this flexibility is to let operations take values in a powerset, and to work with the induced set-level algebra [1, 2, 3]. Because of this

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set-valued viewpoint, hyperstructures have been studied in diverse contexts, including both mathematics and chemical modeling [4, 5, 6].

Superhyperstructures push the same philosophy one level (or several levels) further: instead of using only $\text{PWSET}(S)$, they employ iterated powersets $\text{PWSET}^n(S)$ to represent genuinely hierarchical or multi-layered interactions [7]. This additional expressive power has led to a growing literature spanning mathematics, chemistry, and related areas [8, 9]. Well-known examples include superhypergraph-type constructions [10, 11].

We begin by recalling the iterated powerset notation that underlies these frameworks.

Definition 1 (Base Set). A *base set* S is the underlying collection of elementary objects from which the subsequent powerset and iterated-powerset constructions are formed. Formally,

$$S = \{x \mid x \text{ belongs to the specified universe of discourse}\}.$$

Every element appearing in $\text{PWSET}(S)$ or in $\text{PWSET}^n(S)$ ultimately traces back to elements of S .

Definition 2 (Powerset). [12] For a set S , its *powerset* is the family of all subsets of S :

$$\text{PWSET}(S) = \{A \mid A \subseteq S\}.$$

In particular, $\emptyset \in \text{PWSET}(S)$ and $S \in \text{PWSET}(S)$.

Definition 3 (n -th Powerset). (cf. [7]) For a set H , define the iterated powersets recursively by

$$\text{PWSET}^1(H) = \text{PWSET}(H), \quad \text{PWSET}^{n+1}(H) = \text{PWSET}(\text{PWSET}^n(H)) \quad (n \geq 1).$$

Equivalently, writing $\text{PWSET}_n(H)$ for $\text{PWSET}^n(H)$, we have

$$\text{PWSET}_1(H) = \text{PWSET}(H), \quad \text{PWSET}_{n+1}(H) = \text{PWSET}(\text{PWSET}_n(H)) \quad (n \geq 1).$$

The nonempty variant $\text{PWSET}_n^*(H)$ is defined by

$$\text{PWSET}_1^*(H) = \text{PWSET}(H) \setminus \{\emptyset\}, \quad \text{PWSET}_{n+1}^*(H) = \text{PWSET}(\text{PWSET}_n^*(H)) \setminus \{\emptyset\} \quad (n \geq 1).$$

Example 4 (2-fold powerset in medication regimen planning). Let

$$U = \{\text{Aspirin}, \text{Statin}, \text{ACEi}\}$$

be a base set of individual drugs. A member of $\text{PWSET}(U)$ can be viewed as a *candidate regimen*, for instance

$$\{\text{Aspirin}\}, \quad \{\text{Aspirin}, \text{Statin}\}, \quad \{\text{Statin}, \text{ACEi}\}.$$

An element of the 2-fold powerset $\text{PWSET}^2(U) = \text{PWSET}(\text{PWSET}(U))$ is then a *set of candidate regimens*, i.e., a menu of alternatives. For example,

$$\mathbb{R} = \{\{\text{Aspirin}\}, \{\text{Aspirin}, \text{Statin}\}, \{\text{Statin}, \text{ACEi}\}\} \in \text{PWSET}^2(U)$$

collects three acceptable options for a given profile. Selecting one element of $\mathbb{R}$ fixes a concrete regimen, while retaining $\mathbb{R}$ preserves a portfolio of medically admissible choices for shared decision-making.

To keep the exposition self-contained, we also restate the standard algebraic notions that lead from classical operations to hyper- and superhyper-operations.

Definition 5 (Classical Structure). (cf. [7]) A *classical structure* consists of a nonempty set H equipped with one or more ordinary (single-valued) operations satisfying prescribed axioms. A typical m -ary operation has the form

$$\#_0 : H^m \rightarrow H \quad (m \geq 1).$$

Groups, rings, and fields are familiar instances of this scheme.

Definition 6 (Hyperoperation). (cf. [13, 14]) A *hyperoperation* on a set S is a map

$$\circ : S \times S \longrightarrow \text{PWSET}(S),$$

so that the combination of two inputs yields a subset of S (possibly with more than one element).

Definition 7 (Hyperstructure). (cf. [7, 15, 16]) Given a base set S with a hyperoperation on S (and its induced action on subsets), a corresponding *hyperstructure* may be presented as

$$\mathcal{H} = (\text{PWSET}(S), \circ),$$

where $\circ$ acts at the set level (e.g., for $A, B \subseteq S$, one sets $A \circ B := \bigcup_{a \in A, b \in B} (a \circ b)$).

Example 8 (Hyperstructure for shipping time aggregation with uncertainty). Let $S = \mathbb{N}$ represent business days. For promised legs $x, y \in S$, model a ± 1 day fluctuation by

$$x \diamond y := \{x + y - 1, x + y, x + y + 1\} \subseteq S.$$

Extend this to subsets $A, B \subseteq S$ by

$$A \diamond B := \bigcup_{a \in A, b \in B} (a \diamond b) \subseteq S.$$

Then $\mathcal{H} = (\text{PWSET}(S), \diamond)$ is a hyperstructure: combining two *sets of feasible leg times* produces the *set of feasible end-to-end times*. For $A = \{2, 3\}$ and $B = \{1, 2\}$,

$$\begin{aligned} 2 \diamond 1 &= \{2, 3, 4\}, & 2 \diamond 2 &= \{3, 4, 5\}, \\ 3 \diamond 1 &= \{3, 4, 5\}, & 3 \diamond 2 &= \{4, 5, 6\}, \end{aligned} \quad \Rightarrow \quad A \diamond B = \{2, 3, 4, 5, 6\}.$$

This illustrates how set-valued uncertainty accumulates when legs are concatenated.

Definition 9 (SuperHyperOperation). [7] Let H be nonempty. For $k \geq 0$, define iterated powersets by

$$\text{PWSET}^0(H) = H, \quad \text{PWSET}^{k+1}(H) = \text{PWSET}(\text{PWSET}^k(H)).$$

Fix integers $m, n \geq 0$, an arity $s \geq 1$, and consider inputs from $(\text{PWSET}^m(H))^s$. An (m, n) -*SuperHyperOperation* is a map

$$\odot^{(m,n)} : (\text{PWSET}^m(H))^s \longrightarrow \text{PWSET}^n(H).$$

Allowing $\emptyset$ in the codomain yields the neutrosophic variant; excluding $\emptyset$ corresponds to the classical (nonempty) setting.

Definition 10 (n -Superhyperstructure). (cf. [7, 17]) An n -*Superhyperstructure* extends the hyperstructure idea by moving from $\text{PWSET}(S)$ to the n -fold iterated powerset:

$$\mathcal{SH}_n = (\text{PWSET}^n(S), \circ),$$

where $\circ$ is an operation defined on $\text{PWSET}^n(S)$.

Example 11 (Plant-wide steady-state synthesis as a 2-Superhyperstructure). Let S be the set of feasible steady thermodynamic states of unit operations (pumps, turbines, heat-exchangers, reactors) in a plant. Each $s \in S$ carries valuations

$$\nu_U(s) = \dot{W}(s), \quad \nu_Q(s) = \dot{Q}_{\text{in}}(s), \quad \nu_S(s) = \dot{S}_{\text{gen}}(s) \geq 0,$$

and satisfies the usual steady balances

$$\dot{W} = \dot{Q}_{\text{in}} - \dot{Q}_{\text{out}} - \sum_{\text{streams}} \dot{m} (h_{\text{out}} - h_{\text{in}}), \quad \dot{S}_{\text{gen}} = \sum_k \frac{\dot{Q}_k}{T_k} + \sum_{\text{streams}} \dot{m} (s_{\text{out}} - s_{\text{in}}) \geq 0.$$

Consider $\text{PWSET}^2(S) = \text{PWSET}(\text{PWSET}(S))$. A point $\mathbb{X} \in \text{PWSET}^2(S)$ can be interpreted as a *collection of feasible state-sets* (for example, one feasible set per scenario). Define, for $\mathbb{X}, \mathbb{Y} \in \text{PWSET}^2(S)$,

$$\mathbb{X} \circ \mathbb{Y} := \left\{ \mathbb{Z} \subseteq S \mid \exists X \in \mathbb{X}, Y \in \mathbb{Y} \text{ s.t. } \mathbb{Z} = \{z \in S \mid z \text{ interconnects some } x \in X \text{ and } y \in Y, \right.$$

$$\left. \text{and } z \text{ satisfies node-wise mass/energy balances and } \dot{S}_{\text{gen}}(z) \geq 0 \right\}.$$

Then $\mathcal{SH}_2 = (\text{PWSET}^2(S), \circ)$ is a 2-Superhyperstructure describing meta-ensembles of steady plant states across families of scenarios: the operation preserves the First/Second Law constraints while lifting uncertainty from level 1 to level 2.

Example 12 (Two-phase equilibrium meta-ensembles (phase-splitting) as a 2-Superhyperstructure). Let S be the set of isothermal–isobaric equilibrium states $s = (T, P, x^\ell, y^v)$ for a mixture, where $\mu_i^\ell(T, P, x^\ell) = \mu_i^v(T, P, y^v)$ and

$$z = \lambda x^\ell + (1 - \lambda) y^v \quad (\lambda \in [0, 1])$$

is the overall composition. Work on $\text{PWSET}^2(S)$. An element $\mathbb{F} \in \text{PWSET}^2(S)$ may represent a family of feed-sets (e.g., different trays or feed qualities). Define $\circ$ on $\text{PWSET}^2(S)$ by requiring $\mathbb{Z} \in \mathbb{F} \circ \mathbb{G}$ iff there exist $F \in \mathbb{F}$ and $G \in \mathbb{G}$ such that $\mathbb{Z}$ is exactly the set of all equilibrium splits obtained by pairing one element of F with one element of G while satisfying material/energy balances and the phase rule. Then $(\text{PWSET}^2(S), \circ)$ forms a 2-Superhyperstructure organizing multi-feed or multi-tray uncertainty through a second-level aggregation.

Definition 13 (SuperHyperStructure of order (m, n)). (cf. [8]) Let S be nonempty and let $m, n \geq 0$. A (m, n) -SuperHyperStructure of arity s is specified by a choice of

$$\odot^{(m, n)} : (\text{PWSET}^m(S))^s \rightarrow \text{PWSET}^n(S).$$

This template includes the usual settings as special cases: $(m, n) = (0, 0)$ gives ordinary s -ary operations, $(m, n) = (0, 1)$ recovers hyperoperations, and $s = 1$ corresponds to superhyperfunctions.

Example 14 (Heat-exchanger network synthesis under inlet uncertainty: $(m, n) = (1, 1)$). Let S be the set of two-stream heat-exchanger outlet states

$$s = (T_{h,\text{out}}, T_{c,\text{out}}; T_{h,\text{in}}, T_{c,\text{in}}, \dot{m}_h, \dot{m}_c, c_{p,h}, c_{p,c}),$$

subject to energy balance and a minimum approach constraint:

$$\dot{m}_h c_{p,h} (T_{h,\text{in}} - T_{h,\text{out}}) = \dot{m}_c c_{p,c} (T_{c,\text{out}} - T_{c,\text{in}}), \quad T_{h,\text{out}} - T_{c,\text{in}} \geq \Delta T_{\min}.$$

The entropy generation rate satisfies

$$\dot{S}_{\text{gen}} = \dot{m}_h c_{p,h} \ln \frac{T_{h,\text{in}}}{T_{h,\text{out}}} + \dot{m}_c c_{p,c} \ln \frac{T_{c,\text{out}}}{T_{c,\text{in}}} \geq 0.$$

Let the input space be $(\text{PWSET}^1(S))^2 = (\text{PWSET}(S) \setminus \{\emptyset\})^2$: X_H is a set of admissible hot-side inlet states and X_C a set of admissible cold-side inlet states. Define an (m, n) -superhyperoperation with $(m, n) = (1, 1)$ by

$$\odot^{(1,1)}(X_H, X_C) := \left\{ u \in S \mid \exists x_h \in X_H, x_c \in X_C : u \text{ is consistent with } x_h, x_c, \dot{S}_{\text{gen}}(u) \geq 0 \right\}.$$

Then $(S, \odot^{(1,1)})$ is a SuperHyperStructure of order $(1, 1)$: it maps inlet uncertainty sets to the set of thermodynamically admissible outlet states while enforcing the First/Second Law and the approach constraint.

Example 15 (Robust heat-engine design from reservoir sets: $(m, n) = (1, 2)$). Let S be the set of cycle states $s = (T_h, T_c, \dot{Q}_{\text{in}}, \dot{W}, \eta)$ with efficiency $\eta = \dot{W}/\dot{Q}_{\text{in}}$. Admissibility requires the Carnot bound and nonnegative entropy generation:

$$0 \leq \eta \leq 1 - \frac{T_c}{T_h}, \quad \dot{S}_{\text{gen}} = \frac{\dot{Q}_{\text{in}}}{T_h} - \frac{\dot{Q}_{\text{out}}}{T_c} \geq 0.$$

Let $X \in \text{PWSET}^1(S)$ collect hot-reservoir candidates (e.g., possible T_h values) and let $Y \in \text{PWSET}^1(S)$ collect cold-reservoir candidates (e.g., ambient variability). Define a $(1, 2)$ -superhyperoperation (arity 2) by

$$\odot^{(1,2)}(X, Y) := \left\{ \mathcal{U} \subseteq S \mid \exists X' \subseteq X, Y' \subseteq Y \text{ finite s.t. } \mathcal{U} = \{ u \in S \mid \exists x \in X', y \in Y' : u \text{ pairs } x, y, \right. \\ \left. 0 \leq \eta(u) \leq 1 - \frac{T_c(y)}{T_h(x)}, \dot{S}_{\text{gen}}(u) \geq 0 \right\}.$$

Hence $(S, \odot^{(1,2)})$ is a SuperHyperStructure of order $(1, 2)$: it sends *sets of reservoir states* to *families of feasible cycles*, lifting uncertainty from the reservoir level to a higher-level collection of admissible engine behaviors while preserving the thermodynamic bounds.

For reference, an overview of (classical) structures, hyperstructures, and superhyperstructures is given in Table 1.

TABLE 1. Concise overview of (classical) structures, hyperstructures, and superhyperstructures.

Notion	Formal data (typical)	Operational meaning (one sentence)
(Classical) Structure	$(S, \#_0)$ with an r -ary operation $\#_0 : S^r \rightarrow S$ ($r \geq 1$)	Deterministic composition: combining r inputs yields a single output in S (e.g., groups, rings, vector spaces).
Hyperstructure	$(S, \circ)$ with a hyperoperation $\circ : S \times S \rightarrow \text{PWSET}(S)$ (often extended to $\text{PWSET}(S)$ by unions)	Set-valued composition: combining inputs yields a set of possible outputs, encoding multi-outcome behavior or uncertainty.
Superhyperstructure	$(S, \odot^{(m,n)})$ with an (m, n) -superhyperoperation $\odot^{(m,n)} : (\text{PWSET}^m(S))^r \rightarrow \text{PWSET}^n(S)$	Multi-level composition: inputs/outputs may live in iterated powersets, enabling hierarchical (meta-level) set-valued interactions.

Note. $\text{PWSET}^0(S) = S$ and $\text{PWSET}^{k+1}(S) = \text{PWSET}(\text{PWSET}^k(S))$ for $k \geq 0$; thus $\text{PWSET}^1(S) = \text{PWSET}(S)$ and $\text{PWSET}^n(S)$ is the n -fold iterated powerset.

2 | Main Results

This section states the main contributions of the paper. Our focus is to adapt the notions of *hyperstructures* and *superhyperstructures* to a physical setting and to formalize the basic principles they should satisfy. For reference, we present a concise overview of Physical Structure, Physical HyperStructure, and Physical SuperHyperStructure in Table 2.

TABLE 2. Concise overview of Physical Structure, Physical HyperStructure, and Physical SuperHyperStructure.

Notion	State level	Operation (typical form)	Physical requirements (concise)
Physical Structure	S_{phy} (single states)	deterministic $\sigma_0 : S_{\text{phy}}^{\text{ar}(\sigma)} \rightarrow S_{\text{phy}}$	Feasibility: $\vec{s} \in D_\sigma \Rightarrow \sigma_0(\vec{s})$ is defined. Admissibility: $\nu(\sigma_0(\vec{s})) \in \nu(s_1) + \dots + \nu(s_{\text{ar}(\sigma)}) + C_\sigma$ (often $C_\sigma = \{0\}$). Covariance: $\sigma_0(g \cdot s_1, \dots, g \cdot s_{\text{ar}(\sigma)}) = g \cdot \sigma_0(s_1, \dots, s_{\text{ar}(\sigma)})$.
Physical HyperStructure	S_{phy} (single states, multi-outcomes)	set-valued $\sigma^S : S_{\text{phy}}^{\text{ar}(\sigma)} \rightarrow \text{PWSET}(S_{\text{phy}})$	Feasible nonemptiness: $\vec{s} \in D_\sigma \Rightarrow \sigma^S(\vec{s}) \neq \emptyset$. Conservation: $\forall u \in \sigma^S(\vec{s}), \nu(u) \in \sum_i \nu(s_i) + C_\sigma$. Covariance: $\sigma^S(g \cdot \vec{s}) = g \cdot \sigma^S(\vec{s})$ (set action).
Physical SuperHyperStructure	$\text{PWSET}^m(S_{\text{phy}})$ to $\text{PWSET}^n(S_{\text{phy}})$ (hierarchical sets)	$\odot_\sigma^{(m,n)} : (\text{PWSET}^m(S_{\text{phy}}))^{\text{ar}(\sigma)} \rightarrow \text{PWSET}^n(S_{\text{phy}})$	Feasible nonemptiness via atoms: if $\exists s_i \in \text{At}(X_i)$ with $(s_1, \dots, s_{\text{ar}(\sigma)}) \in D_\sigma$, then $\odot_\sigma^{(m,n)}(\vec{X}) \neq \emptyset$. Atomic admissibility: for $Y \in \odot_\sigma^{(m,n)}(\vec{X})$ and $u \in \text{At}(Y)$, $\exists s_i \in \text{At}(X_i)$ with $\nu(u) \in \sum_i \nu(s_i) + C_\sigma$. Lifted covariance: $\odot_\sigma^{(m,n)}(g^{\uparrow m} \cdot \vec{X}) = g^{\uparrow n} \cdot \odot_\sigma^{(m,n)}(\vec{X})$.

Note. Setting $C_\sigma = \{0\}$ enforces strict conservation. The scheme interpolates levels: $(m, n) = (0, 0)$ gives ordinary (single-outcome) operations, and $(m, n) = (0, 1)$ recovers the Physical HyperStructure case.

2.1 | Physical HyperStructure

Informally, a Physical HyperStructure treats physical states as the underlying set and models interactions by set-valued operations. The admissible outputs are required to respect feasibility constraints, conservation laws (possibly with controlled losses), and covariance under a chosen symmetry group.

Definition 16 (Physical HyperStructure). Let S_{phy} be a nonempty set of *physical states*. Let $(V, +)$ be a commutative monoid (for instance $V = \mathbb{R}^d$ with vector addition) used to encode extensive/kinematic quantities such as momentum, energy, charge, and related invariants. Let $\nu : S_{\text{phy}} \rightarrow V$ be a *physical valuation* assigning to each state its relevant quantity vector.

Let G be a symmetry group (e.g., Galilean, Poincaré, rotation, etc.) acting on S_{phy} , and let $\rho : G \rightarrow \text{Aut}(V)$ be a representation such that

$$\nu(g \cdot s) = \rho(g) \nu(s) \quad (g \in G, s \in S_{\text{phy}}).$$

Fix a signature Σ_{phy} of interaction symbols with arities $\text{ar}(\sigma) \geq 1$. For each $\sigma \in \Sigma_{\text{phy}}$, choose

- a *feasibility domain* $D_\sigma \subseteq S_{\text{phy}}^{\text{ar}(\sigma)}$ (encoding kinematic/local constraints),
- a closed convex cone $C_\sigma \subseteq V$ (encoding admissible net losses/gains; the choice $C_\sigma = \{0\}$ enforces strict conservation).

A *Physical HyperStructure* is a tuple

$$\mathbf{PHS} = (S_{\text{phy}}, \Sigma_{\text{phy}}, \{\sigma^S\}_{\sigma \in \Sigma_{\text{phy}}}, \nu, G, \{C_\sigma\}_{\sigma \in \Sigma_{\text{phy}}}),$$

where each *interaction hyperoperation*

$$\sigma^S : S_{\text{phy}}^{\text{ar}(\sigma)} \longrightarrow \text{PWSET}(S_{\text{phy}})$$

satisfies the following properties for every input $\vec{s} = (s_1, \dots, s_{\text{ar}(\sigma)})$:

- Feasible nonemptiness*: if $\vec{s} \in D_\sigma$, then $\sigma^S(\vec{s}) \neq \emptyset$ (for $\vec{s} \notin D_\sigma$ the value may be empty).
- Cone-conservation (physical admissibility)*: for every $u \in \sigma^S(\vec{s})$,

$$\nu(u) \in \nu(s_1) + \dots + \nu(s_{\text{ar}(\sigma)}) + C_\sigma.$$

- Symmetry covariance*: for every $g \in G$ and every $\vec{s} \in S_{\text{phy}}^{\text{ar}(\sigma)}$,

$$\sigma^S(g \cdot s_1, \dots, g \cdot s_{\text{ar}(\sigma)}) = g \cdot \sigma^S(s_1, \dots, s_{\text{ar}(\sigma)}).$$

As usual in hyperstructure theory, when $\text{ar}(\sigma) = 2$ the induced operation on subsets is

$$A \sigma B := \bigcup_{a \in A, b \in B} \sigma^S(a, b) \quad (A, B \subseteq S_{\text{phy}}).$$

Example 17 (Binary collisions with restitution). Let

$$S_{\text{phy}} = \{(m, \mathbf{v}) \mid m > 0, \mathbf{v} \in \mathbb{R}^d\}.$$

Take $V = \mathbb{R}^d \times \mathbb{R}_{\geq 0}$ with componentwise addition, and set

$$\nu(m, \mathbf{v}) = \left(m\mathbf{v}, \frac{1}{2}m\|\mathbf{v}\|^2\right)$$

(total momentum, kinetic energy). For $\sigma = \text{coll}$ (arity 2), choose $D_\sigma = S_{\text{phy}}^2$ and

$$C_\sigma = \{0\}^d \times [0, \Delta E],$$

allowing up to ΔE of kinetic-energy loss. Define $\sigma^S((m_1, \mathbf{v}_1), (m_2, \mathbf{v}_2))$ as the set of all post-collision pairs consistent with exact momentum conservation and energy loss bounded by ΔE . Let $G = \text{SE}(d)$ act by rigid motions on velocities, with ρ the induced linear action on V . Then the covariance requirement is satisfied, and the cone condition precisely encodes bounded dissipation.

Example 18 (Quantum projective measurement with bounded back-action). (cf.[18, 19]) Let $\mathcal{H}$ be a finite-dimensional Hilbert space with Hamiltonian H . Let S_{phy} be the set of density matrices on $\mathcal{H}$. Take $V = \mathbb{R}$ (addition) and define

$$\nu(\rho) := \text{Tr}(H\rho) \quad (\text{energy expectation}).$$

Let

$$G = \{U \in \text{U}(\mathcal{H}) \mid [U, H] = 0\}$$

act by conjugation $U \cdot \rho := U\rho U^\dagger$, and take $\rho(g) = \text{id}_{\mathbb{R}}$, so $\nu(U \cdot \rho) = \nu(\rho)$. Fix a projective measurement $\Pi = \{P_i\}_{i \in I}$ with $P_i P_j = \delta_{ij} P_i$ and $\sum_i P_i = I$. Let $\sigma = \text{meas}_\Pi$ have arity 1, feasibility domain $D_\sigma = S_{\text{phy}}$, and cone

$$C_\sigma = [-\Delta E, \Delta E] \subset \mathbb{R}$$

representing bounded energy back-action. Define $\sigma^S : S_{\text{phy}} \rightarrow \text{PWSET}(S_{\text{phy}})$ by

$$\sigma^S(\rho) := \left\{ \rho_i = \frac{P_i \rho P_i}{\text{Tr}(P_i \rho)} \mid \text{Tr}(P_i \rho) > 0, i \in I \right\}.$$

Then: (i) the output is nonempty whenever some $\text{Tr}(P_i \rho) > 0$ (and this is precisely the feasible case since $D_\sigma = S_{\text{phy}}$); (ii) for each outcome $u = \rho_i$,

$$\nu(u) - \nu(\rho) = \text{Tr}(H(\rho_i - \rho)) \in C_\sigma,$$

which becomes 0 whenever $[H, P_i] = 0$; (iii) covariance holds because U commutes with H :

$$\sigma^S(U \cdot \rho) = \{ U \rho_i U^\dagger \mid \rho_i \in \sigma^S(\rho) \} = U \cdot \sigma^S(\rho).$$

Hence $(S_{\text{phy}}, \sigma^S, \nu, G, C_\sigma)$ satisfies Definition 16.

Example 19 (Relativistic two-body decay with exact four-momentum conservation). Let S_{phy} be the set of finite particle configurations $X = \{p_1, \dots, p_k\}$ where each $p_\ell \in \mathbb{R}^{1,3}$ lies on the mass shell ($p_\ell^2 = m_\ell^2$ and $p_\ell^0 > 0$). Let $V = \mathbb{R}^{1,3}$ with vector addition and define

$$\nu(X) := \sum_{p \in X} p \quad (\text{total four-momentum}).$$

Let G be the Poincaré group acting diagonally on configurations, and let $\rho(g)$ be its standard linear action on V , so $\nu(g \cdot X) = \rho(g)\nu(X)$.

For a parent particle with four-momentum p (viewed as $X = \{p\}$), consider the unary interaction $\sigma = \text{decay}_{m_1, m_2}$ with feasibility domain

$$D_\sigma = \{ \{p\} \mid p^2 \geq (m_1 + m_2)^2 \}, \quad C_\sigma = \{0\} \subset \mathbb{R}^{1,3}.$$

Define

$$\sigma^S(\{p\}) := \{ \{q, r\} \mid q^2 = m_1^2, r^2 = m_2^2, q^0 > 0, r^0 > 0, q + r = p \}.$$

If $\{p\} \in D_\sigma$, standard two-body kinematics yields a nonempty solution set. Moreover, for any $u = \{q, r\}$ we have $\nu(u) = q + r = p = \nu(\{p\})$, hence strict conservation holds, and covariance is immediate from Poincaré equivariance of the defining relations.

Example 20 (Rigid impact with Coulomb friction and restitution). (cf.[20, 21]) Let S_{phy} consist of two-body states $(m_1, \mathbf{v}_1; m_2, \mathbf{v}_2)$ with $\mathbf{v}_i \in \mathbb{R}^d$. Let $V = \mathbb{R}^d \times \mathbb{R}_{\geq 0}$ and set

$$\nu(m_1, \mathbf{v}_1; m_2, \mathbf{v}_2) = \left(m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2, \frac{1}{2} m_1 \|\mathbf{v}_1\|^2 + \frac{1}{2} m_2 \|\mathbf{v}_2\|^2 \right).$$

Let $G = \text{SE}(d)$ act by rigid motions on velocities (trivial on masses) and via the induced linear action on V .

Fix contact geometry: a unit normal $\mathbf{n}$ and a tangent frame for admissible tangential directions. Let $\mu \geq 0$ be a Coulomb coefficient and $e \in [0, 1]$ a restitution parameter. Consider the binary interaction $\sigma = \text{impact}_{\mu, e}$ with $D_\sigma = S_{\text{phy}}^2$ and

$$C_\sigma = \{0\}^d \times [0, \Delta E],$$

allowing at most ΔE of kinetic-energy loss.

For $\vec{s} = ((m_1, \mathbf{v}_1), (m_2, \mathbf{v}_2))$, define the admissible impulse cone

$$\mathcal{J} = \left\{ \mathbf{J} = \lambda \mathbf{n} + \mathbf{t} \mid \lambda \geq 0, \|\mathbf{t}\| \leq \mu \lambda \right\},$$

and define post-impact velocities by

$$\mathbf{v}'_1 = \mathbf{v}_1 + \frac{\mathbf{J}}{m_1}, \quad \mathbf{v}'_2 = \mathbf{v}_2 - \frac{\mathbf{J}}{m_2}.$$

Impose the restitution constraint on the normal component of the relative velocity,

$$-(\mathbf{v}'_1 - \mathbf{v}'_2) \cdot \mathbf{n} \in [e(\mathbf{v}_1 - \mathbf{v}_2) \cdot \mathbf{n}]_+.$$

Set

$$\sigma^S(\vec{s}) := \{ (m_1, \mathbf{v}'_1; m_2, \mathbf{v}'_2) \mid \mathbf{J} \in \mathcal{J} \text{ satisfies the restitution constraint} \}.$$

Then feasibility (and hence nonemptiness for approaching contacts) follows from standard rigid-impact theory. Momentum is conserved exactly by construction, while the kinetic-energy decrease lies in $[0, \Delta E]$, so

$$\nu(u) - \nu(\vec{s}) \in C_\sigma \quad \forall u \in \sigma^S(\vec{s}).$$

Covariance under rigid motions holds because the contact geometry and cone constraints are defined geometrically.

Example 21 (Fluid parcel mixing with bounded dissipation). (cf.[22]) Let S_{phy} be the set of parcels $s = (m, \mathbf{v}, T)$ with $m > 0$, $\mathbf{v} \in \mathbb{R}^d$, and $T \geq 0$. Let $V = \mathbb{R}^d \times \mathbb{R}_{\geq 0}$ and define

$$\nu(m, \mathbf{v}, T) = \left(m\mathbf{v}, \frac{1}{2}m\|\mathbf{v}\|^2 + cmT \right),$$

where $c > 0$ is a heat capacity (so the second component is mechanical plus internal energy). Let $G = \text{SE}(d)$ act by rigid motions on velocities and trivially on (m, T) .

Consider $\sigma = \text{mix}$ (arity 2) with $D_\sigma = S_{\text{phy}}^2$ and

$$C_\sigma = \{0\}^d \times [0, \Lambda],$$

interpreted as allowing up to Λ of energy transfer to unresolved scales. Given $s_1 = (m_1, \mathbf{v}_1, T_1)$ and $s_2 = (m_2, \mathbf{v}_2, T_2)$, define

$$\sigma^S(s_1, s_2) := \left\{ (m_1 + m_2, \mathbf{v}, T) \mid m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = (m_1 + m_2)\mathbf{v}, \nu(s_1) + \nu(s_2) - \nu(m_1 + m_2, \mathbf{v}, T) \in \{0\}^d \times [0, \Lambda] \right\}.$$

Nonemptiness is obtained by choosing $\mathbf{v}$ to satisfy momentum conservation and then selecting T so that the energy loss falls in $[0, \Lambda]$. The cone-admissibility condition holds by construction, and covariance under rigid motions is immediate. This example captures coarse-grained mixing with controlled dissipation.

Proposition 22 (Lifted G -action on $\text{PWSET}(S_{\text{phy}})$). *Let G act on S_{phy} . The rule $g \cdot A := \{g \cdot a \mid a \in A\}$ defines a left action of G on $\text{PWSET}(S_{\text{phy}})$, i.e.*

$$e \cdot A = A, \quad (gh) \cdot A = g \cdot (h \cdot A) \quad (g, h \in G, A \subseteq S_{\text{phy}}).$$

Proof: For the identity element $e \in G$,

$$e \cdot A = \{e \cdot a \mid a \in A\} = \{a \mid a \in A\} = A.$$

For $g, h \in G$,

$$(gh) \cdot A = \{(gh) \cdot a \mid a \in A\} = \{g \cdot (h \cdot a) \mid a \in A\}.$$

On the other hand,

$$g \cdot (h \cdot A) = g \cdot \{h \cdot a \mid a \in A\} = \{g \cdot x \mid x \in \{h \cdot a \mid a \in A\}\} = \{g \cdot (h \cdot a) \mid a \in A\}.$$

Thus $(gh) \cdot A = g \cdot (h \cdot A)$. $\square$

Proposition 23 (Powerset extension preserves covariance). *Let $\sigma \in \Sigma_{\text{phy}}$ have arity r , and define the induced set-valued operation on subsets by*

$$\bar{\sigma}(A_1, \dots, A_r) := \bigcup_{a_1 \in A_1, \dots, a_r \in A_r} \sigma^S(a_1, \dots, a_r) \quad (A_i \subseteq S_{\text{phy}}).$$

Then for every $g \in G$ and every $A_1, \dots, A_r \subseteq S_{\text{phy}}$,

$$\bar{\sigma}(g \cdot A_1, \dots, g \cdot A_r) = g \cdot \bar{\sigma}(A_1, \dots, A_r).$$

In particular, when $r = 2$ this reads $(g \cdot A) \sigma (g \cdot B) = g \cdot (A \sigma B)$.

Proof: Fix $g \in G$. By definition of $\bar{\sigma}$,

$$\bar{\sigma}(g \cdot A_1, \dots, g \cdot A_r) = \bigcup_{x_1 \in g \cdot A_1, \dots, x_r \in g \cdot A_r} \sigma^S(x_1, \dots, x_r).$$

Each $x_i \in g \cdot A_i$ has the form $x_i = g \cdot a_i$ for some $a_i \in A_i$, hence

$$\bar{\sigma}(g \cdot A_1, \dots, g \cdot A_r) = \bigcup_{a_1 \in A_1, \dots, a_r \in A_r} \sigma^S(g \cdot a_1, \dots, g \cdot a_r).$$

Using symmetry covariance from Definition 16(iii),

$$\sigma^S(g \cdot a_1, \dots, g \cdot a_r) = g \cdot \sigma^S(a_1, \dots, a_r).$$

Therefore,

$$\bar{\sigma}(g \cdot A_1, \dots, g \cdot A_r) = \bigcup_{a_1 \in A_1, \dots, a_r \in A_r} g \cdot \sigma^S(a_1, \dots, a_r) = g \cdot \bigcup_{a_1 \in A_1, \dots, a_r \in A_r} \sigma^S(a_1, \dots, a_r) = g \cdot \bar{\sigma}(A_1, \dots, A_r).$$

□

Proposition 24 (Monotonicity and union-distributivity). *Let $\sigma \in \Sigma_{\text{phy}}$ have arity r and let $\bar{\sigma}$ be as in Proposition 23. Then for any $j \in \{1, \dots, r\}$:*

(i) *If $A_j \subseteq A'_j$ and A_i are fixed for $i \neq j$, then*

$$\bar{\sigma}(A_1, \dots, A_j, \dots, A_r) \subseteq \bar{\sigma}(A_1, \dots, A'_j, \dots, A_r).$$

(ii) *For any family $\{B_\lambda\}_{\lambda \in \Lambda} \subseteq \text{PWSET}(S_{\text{phy}})$,*

$$\bar{\sigma}(A_1, \dots, \bigcup_{\lambda \in \Lambda} B_\lambda, \dots, A_r) = \bigcup_{\lambda \in \Lambda} \bar{\sigma}(A_1, \dots, B_\lambda, \dots, A_r).$$

Proof: (i) If $A_j \subseteq A'_j$, then every tuple $(a_1, \dots, a_r)$ with $a_j \in A_j$ is also a tuple with $a_j \in A'_j$. Hence, taking unions over a larger index set can only enlarge the result:

$$\bigcup_{a_j \in A_j} \dots \subseteq \bigcup_{a_j \in A'_j} \dots,$$

which is exactly the desired inclusion.

(ii) By definition,

$$\bar{\sigma}(A_1, \dots, \bigcup_{\lambda} B_\lambda, \dots, A_r) = \bigcup_{b \in \bigcup_{\lambda} B_\lambda} \bigcup_{a_i \in A_i, i \neq j} \sigma^S(a_1, \dots, b, \dots, a_r).$$

Since $\bigcup_{\lambda} B_\lambda$ is a disjoint-union-free union of indices,

$$\bigcup_{b \in \bigcup_{\lambda} B_\lambda} (\dots) = \bigcup_{\lambda \in \Lambda} \bigcup_{b \in B_\lambda} (\dots) = \bigcup_{\lambda \in \Lambda} \bar{\sigma}(A_1, \dots, B_\lambda, \dots, A_r).$$

□

Theorem 25 (Valuation bound for outputs (cone-conservation as a set inclusion)). *Let $\sigma \in \Sigma_{\text{phy}}$ have arity r and fix $(s_1, \dots, s_r) \in S_{\text{phy}}^r$. Then*

$$\nu(\sigma^S(s_1, \dots, s_r)) \subseteq \{\nu(s_1)\} + \dots + \{\nu(s_r)\} + C_\sigma.$$

More generally, for subsets $A_1, \dots, A_r \subseteq S_{\text{phy}}$,

$$\nu(\bar{\sigma}(A_1, \dots, A_r)) \subseteq \nu(A_1) + \dots + \nu(A_r) + C_\sigma.$$

In the binary case this reads $\nu(A \sigma B) \subseteq \nu(A) + \nu(B) + C_\sigma$.

Proof: For the pointwise statement, take any $w \in \nu(\sigma^S(s_1, \dots, s_r))$. Then $w = \nu(u)$ for some $u \in \sigma^S(s_1, \dots, s_r)$. By Definition 16(ii),

$$\nu(u) \in \nu(s_1) + \dots + \nu(s_r) + C_\sigma = \{\nu(s_1)\} + \dots + \{\nu(s_r)\} + C_\sigma.$$

Hence w lies in the right-hand side, proving the inclusion.

For the subset statement, take any $w \in \nu(\bar{\sigma}(A_1, \dots, A_r))$. Then $w = \nu(u)$ for some

$$u \in \bar{\sigma}(A_1, \dots, A_r) = \bigcup_{a_1 \in A_1, \dots, a_r \in A_r} \sigma^S(a_1, \dots, a_r).$$

Thus there exist $a_i \in A_i$ such that $u \in \sigma^S(a_1, \dots, a_r)$. Applying the pointwise inclusion to $(a_1, \dots, a_r)$ yields

$$\nu(u) \in \{\nu(a_1)\} + \dots + \{\nu(a_r)\} + C_\sigma \subseteq \nu(A_1) + \dots + \nu(A_r) + C_\sigma,$$

since $\nu(a_i) \in \nu(A_i)$. Therefore w lies in the right-hand side, proving the inclusion. □

Theorem 26 (Residual (defect) set and its covariance). *Assume $\sigma \in \Sigma_{\text{phy}}$ has arity r . For $\vec{s} = (s_1, \dots, s_r) \in S_{\text{phy}}^r$ define the residual set*

$$\Delta_\sigma(\vec{s}) := \left\{ c \in V \mid \exists u \in \sigma^S(\vec{s}) \text{ such that } \nu(u) = \nu(s_1) + \dots + \nu(s_r) + c \right\} \subseteq V.$$

Then:

(i) $\Delta_\sigma(\vec{s}) \subseteq C_\sigma$ for all $\vec{s}$.

(ii) For every $g \in G$,

$$\Delta_\sigma(g \cdot s_1, \dots, g \cdot s_r) = \rho(g) \Delta_\sigma(s_1, \dots, s_r).$$

Proof: (i) Let $c \in \Delta_\sigma(\vec{s})$. Then $\nu(u) = \sum_{i=1}^r \nu(s_i) + c$ for some $u \in \sigma^S(\vec{s})$. By Definition 16(ii),

$$\nu(u) \in \sum_{i=1}^r \nu(s_i) + C_\sigma.$$

Hence there exists $c' \in C_\sigma$ with $\nu(u) = \sum_i \nu(s_i) + c'$. Comparing with $\nu(u) = \sum_i \nu(s_i) + c$ gives $c = c' \in C_\sigma$, so $\Delta_\sigma(\vec{s}) \subseteq C_\sigma$.

(ii) First show $\rho(g)\Delta_\sigma(\vec{s}) \subseteq \Delta_\sigma(g \cdot \vec{s})$. Take $d \in \Delta_\sigma(\vec{s})$, so $\nu(u) = \sum_i \nu(s_i) + d$ for some $u \in \sigma^S(\vec{s})$. By symmetry covariance (Definition 16(iii)), $g \cdot u \in \sigma^S(g \cdot s_1, \dots, g \cdot s_r)$. Applying valuation equivariance $\nu(g \cdot x) = \rho(g)\nu(x)$ and using that $\rho(g)$ is additive,

$$\nu(g \cdot u) = \rho(g)\nu(u) = \rho(g)\left(\sum_{i=1}^r \nu(s_i) + d\right) = \sum_{i=1}^r \rho(g)\nu(s_i) + \rho(g)d = \sum_{i=1}^r \nu(g \cdot s_i) + \rho(g)d.$$

Thus $\rho(g)d \in \Delta_\sigma(g \cdot s_1, \dots, g \cdot s_r)$.

For the reverse inclusion, apply the established inclusion to g^{-1} and then multiply by $\rho(g)$:

$$\rho(g^{-1})\Delta_\sigma(g \cdot \vec{s}) \subseteq \Delta_\sigma(\vec{s}) \implies \Delta_\sigma(g \cdot \vec{s}) \subseteq \rho(g)\Delta_\sigma(\vec{s}).$$

Hence equality holds. $\square$

Corollary 27 (Strict conservation as an equality). *If $C_\sigma = \{0\}$ for some $\sigma \in \Sigma_{\text{phy}}$ of arity r , then for every $\vec{s} = (s_1, \dots, s_r)$ and every $u \in \sigma^S(\vec{s})$,*

$$\nu(u) = \nu(s_1) + \dots + \nu(s_r).$$

Equivalently, $\Delta_\sigma(\vec{s}) = \{0\}$ for all $\vec{s}$.

Proof: If $u \in \sigma^S(\vec{s})$, then by Definition 16(ii),

$$\nu(u) \in \nu(s_1) + \dots + \nu(s_r) + C_\sigma = \nu(s_1) + \dots + \nu(s_r) + \{0\}.$$

Thus $\nu(u) = \nu(s_1) + \dots + \nu(s_r)$. The residual-set statement follows from the definition of Δ_σ . $\square$

Theorem 28 (Sequential composition of unary interactions). *Let $\sigma, \tau \in \Sigma_{\text{phy}}$ be unary symbols (arity 1). Define the composed hyperoperation*

$$(\tau \star \sigma)^S(s) := \bigcup_{x \in \sigma^S(s)} \tau^S(x) \quad (s \in S_{\text{phy}}).$$

Assume that both σ^S and τ^S satisfy Definition 16 with cones C_σ, C_τ . Then $(\tau \star \sigma)^S$ satisfies the cone-conservation and covariance axioms with cone $C_\sigma + C_\tau$, namely:

(i) For every $u \in (\tau \star \sigma)^S(s)$,

$$\nu(u) \in \nu(s) + (C_\sigma + C_\tau).$$

(ii) For every $g \in G$,

$$(\tau \star \sigma)^S(g \cdot s) = g \cdot (\tau \star \sigma)^S(s).$$

If, in addition, $\sigma^S(s) \neq \emptyset$ for all $s \in D_\sigma$ and $\tau^S(x) \neq \emptyset$ for all $x \in D_\tau$, then $(\tau \star \sigma)^S(s) \neq \emptyset$ for every $s \in D_\sigma$ with $\sigma^S(s) \subseteq D_\tau$.

Proof: Take $u \in (\tau \star \sigma)^S(s)$. Then there exists $x \in \sigma^S(s)$ such that $u \in \tau^S(x)$. By cone-conservation for σ^S ,

$$\nu(x) \in \nu(s) + C_\sigma,$$

so $\nu(x) = \nu(s) + c_\sigma$ for some $c_\sigma \in C_\sigma$ (in the ambient additive setting underlying V). By cone-conservation for τ^S ,

$$\nu(u) \in \nu(x) + C_\tau,$$

so $\nu(u) = \nu(x) + c_\tau$ for some $c_\tau \in C_\tau$. Therefore

$$\nu(u) = \nu(s) + c_\sigma + c_\tau \in \nu(s) + (C_\sigma + C_\tau),$$

proving (i).

For (ii), using covariance of σ^S and τ^S ,

$$(\tau \star \sigma)^S(g \cdot s) = \bigcup_{y \in \sigma^S(g \cdot s)} \tau^S(y) = \bigcup_{x \in \sigma^S(s)} \tau^S(g \cdot x) = \bigcup_{x \in \sigma^S(s)} g \cdot \tau^S(x) = g \cdot \bigcup_{x \in \sigma^S(s)} \tau^S(x) = g \cdot (\tau \star \sigma)^S(s).$$

For the nonemptiness statement: if $s \in D_\sigma$ then $\sigma^S(s) \neq \emptyset$. If moreover $\sigma^S(s) \subseteq D_\tau$, then for each $x \in \sigma^S(s)$ we have $\tau^S(x) \neq \emptyset$, so the union $\bigcup_{x \in \sigma^S(s)} \tau^S(x)$ is nonempty. $\square$

2.2 | Physical SuperHyperStructure

A Physical SuperHyperStructure extends the physical hyperstructure viewpoint by allowing inputs and outputs to live in iterated powersets. This makes it possible to encode multi-level collections of states (ensembles, families of ensembles, and so on) while still enforcing feasibility, conservation (possibly with controlled loss), and covariance under symmetries.

Definition 29 (Iterated powersets, atoms, and lifted actions). For $k \geq 0$, define the k -fold iterated powerset of S_{phy} recursively by

$$\text{PWSET}^0((S_{\text{phy}})) = S_{\text{phy}}, \quad \text{PWSET}^{k+1}((S_{\text{phy}})) = \text{PWSET}(\text{PWSET}^k((S_{\text{phy}}))).$$

For $Z \in \text{PWSET}^k((S_{\text{phy}}))$, define its set of *atoms* $\text{At}(Z) \subseteq S_{\text{phy}}$ by recursion:

$$\text{At}(x) = \{x\} \quad (x \in S_{\text{phy}}), \quad \text{At}(Z) = \bigcup_{z \in Z} \text{At}(z) \quad (k \geq 1).$$

If G acts on S_{phy} , then each $g \in G$ acts on $\text{PWSET}^k((S_{\text{phy}}))$ by its canonical lift:

$$g^{\uparrow 0} \cdot x := g \cdot x, \quad g^{\uparrow(k+1)} \cdot X := \{g^{\uparrow k} \cdot x \mid x \in X\}.$$

Definition 30 (Physical SuperHyperStructure of order (m, n)). Fix integers $m, n \geq 0$ and a signature Σ_{phy} of symbols σ with arities $\text{ar}(\sigma) \geq 1$. Assume that, for each $\sigma \in \Sigma_{\text{phy}}$, we are given a feasibility domain $D_\sigma \subseteq S_{\text{phy}}^{\text{ar}(\sigma)}$ and a closed convex cone $C_\sigma \subseteq V$.

A *Physical SuperHyperStructure of order (m, n)* is a tuple

$$\text{PSHS} = (S_{\text{phy}}, \Sigma_{\text{phy}}, \{\odot_\sigma^{(m,n)}\}_{\sigma \in \Sigma_{\text{phy}}}, \nu, G, \{C_\sigma\}_{\sigma \in \Sigma_{\text{phy}}}),$$

where each *superhyperoperation*

$$\odot_\sigma^{(m,n)} : (\text{PWSET}^m((S_{\text{phy}})))^{\text{ar}(\sigma)} \longrightarrow \text{PWSET}^n((S_{\text{phy}}))$$

satisfies the following conditions for every input $\vec{X} = (X_1, \dots, X_{\text{ar}(\sigma)})$:

- (i) *Feasible nonemptiness*: if there exist atoms $s_i \in \text{At}(X_i)$ such that $(s_1, \dots, s_{\text{ar}(\sigma)}) \in D_\sigma$, then $\odot_\sigma^{(m,n)}(\vec{X}) \neq \emptyset$.
- (ii) *Atomic physical admissibility*: for every $Y \in \odot_\sigma^{(m,n)}(\vec{X})$ and every atom $u \in \text{At}(Y)$, there exist $s_i \in \text{At}(X_i)$ such that

$$\nu(u) \in \nu(s_1) + \dots + \nu(s_{\text{ar}(\sigma)}) + C_\sigma.$$

(iii) *Lifted covariance*: for every $g \in G$,

$$\odot_{\sigma}^{(m,n)}(g^{\uparrow m} \cdot X_1, \dots, g^{\uparrow m} \cdot X_{\text{ar}(\sigma)}) = g^{\uparrow n} \cdot \odot_{\sigma}^{(m,n)}(X_1, \dots, X_{\text{ar}(\sigma)}).$$

If $m = 0$ and $n = 1$ and $\odot_{\sigma}^{(0,1)}$ is identified with the interaction hyperoperation σ^S , then Definition 30 specializes to a Physical HyperStructure as in Definition 16.

Example 31 (Ensemble-level collision step with model uncertainty). Assume the collision-based Physical HyperStructure from the binary-collision example, and fix $(m, n) = (1, 2)$. For nonempty $X_1, X_2 \in \text{PWSET}(S_{\text{phy}})$, define

$$\odot_{\text{coll}}^{(1,2)}(X_1, X_2) := \left\{ \mathcal{Y} \subseteq \text{PWSET}(S_{\text{phy}}) \setminus \{\emptyset\} \mid \exists \text{ finite } \mathcal{M} \subseteq [e_{\min}, e_{\max}] \text{ s.t. } \exists Y \in \mathcal{Y} \text{ with } \right. \\ \left. Y = \bigcup_{e \in \mathcal{M}} \bigcup_{\substack{s_1 \in X_1 \\ s_2 \in X_2}} \sigma_e^S(s_1, s_2) \right\},$$

where σ_e^S denotes the collision rule with restitution parameter e . Each $\mathcal{Y}$ represents a collection of admissible post-collision ensembles generated from a finite set of model parameters $\mathcal{M}$. The feasibility, atomic admissibility, and lifted covariance follow from the corresponding properties of σ_e^S together with the definition of $\text{At}(\cdot)$.

Example 32 (Quantum channel uncertainty on ensembles: $(m, n) = (1, 2)$). Let $\mathcal{H}$ be finite-dimensional and let S_{phy} be the set of density matrices on $\mathcal{H}$. Fix a Hamiltonian H and take $V = \mathbb{R}$ with valuation $\nu(\rho) = \text{Tr}(H\rho)$. Let

$$G = \{U \in \text{U}(\mathcal{H}) \mid [U, H] = 0\}$$

act by conjugation $U^{\uparrow 0} \cdot \rho = U\rho U^{\dagger}$, and take the action on V to be trivial, so $\nu(U \cdot \rho) = \nu(\rho)$. Fix a cone $C_{\sigma} = [-\Delta E, \Delta E]$. Let $\mathfrak{M}$ be a nonempty family of CPTP maps (models) that is *closed under G -covariance*: for every $U \in G$ and every $\mathcal{E} \in \mathfrak{M}$ there exists $\mathcal{E}' \in \mathfrak{M}$ such that

$$U\mathcal{E}(\rho)U^{\dagger} = \mathcal{E}'(U\rho U^{\dagger}) \quad (\rho \in S_{\text{phy}}).$$

Define $\odot_{\sigma}^{(1,2)} : \text{PWSET}(S_{\text{phy}}) \setminus \{\emptyset\} \rightarrow \text{PWSET}(\text{PWSET}(S_{\text{phy}}))$ by

$$\odot_{\sigma}^{(1,2)}(X) := \left\{ \mathcal{Y} \subseteq S_{\text{phy}} \mid \exists \text{ finite } \mathcal{M} \subseteq \mathfrak{M} \text{ s.t. } \mathcal{Y} = \{ \mathcal{E}(\rho) \mid \rho \in X, \mathcal{E} \in \mathcal{M} \} \right\}.$$

Checks. (i) Nonemptiness holds since $X \neq \emptyset$ and $\mathcal{M} \neq \emptyset$. (ii) If $Y \in \odot_{\sigma}^{(1,2)}(X)$ and $u = \mathcal{E}(\rho) \in \text{At}(Y)$, then the bounded back-action assumption gives

$$\nu(u) - \nu(\rho) = \text{Tr}(H(\mathcal{E}(\rho) - \rho)) \in C_{\sigma}.$$

(iii) For $U \in G$, the covariance closure of $\mathfrak{M}$ yields the lifted relation

$$\odot_{\sigma}^{(1,2)}(U^{\uparrow 1} \cdot X) = \left\{ U^{\uparrow 2} \cdot Y \mid Y \in \odot_{\sigma}^{(1,2)}(X) \right\}.$$

Thus $(m, n) = (1, 2)$ produces a PSHS that captures model-level channel uncertainty on ensembles.

Example 33 (Many-body time stepping with integrator/model ensembles: $(m, n) = (1, 2)$). Let $S_{\text{phy}} = \mathbb{R}^{3N} \times \mathbb{R}^{3N}$ be the set of N -body states $s = (q, p)$. Let $V = \mathbb{R}^3 \times \mathbb{R}$ and define

$$\nu(q, p) = \left(\sum_{i=1}^N p_i, H(q, p) \right),$$

where H is the Hamiltonian (so ν records total linear momentum and total energy). Let $G = \text{SE}(3)$ act diagonally on (q, p) , and let ρ be the induced linear action on V .

Fix $\Delta t > 0$ and a nonempty family $\mathfrak{S}$ of one-step maps Φ (e.g., symplectic schemes, or schemes with drag $\gamma \in [0, \gamma_{\max}]$). Choose a closed convex cone $C_{\sigma} \subseteq V$ bounding admissible momentum/energy drift over one step, for example $C_{\sigma} = B_{\Pi} \times [-\Delta E, \Delta E]$, where $B_{\Pi} \subseteq \mathbb{R}^3$ is a convex cone of net impulse.

Define $\odot_{\text{step}}^{(1,2)} : \text{PWSET}(S_{\text{phy}}) \setminus \{\emptyset\} \rightarrow \text{PWSET}(\text{PWSET}(S_{\text{phy}}))$ by

$$\odot_{\text{step}}^{(1,2)}(X) := \left\{ \mathcal{Y} \subseteq S_{\text{phy}} \mid \exists \text{ finite } \mathcal{M} \subseteq \mathfrak{S} \text{ s.t. } \mathcal{Y} = \{ \Phi(s) \mid s \in X, \Phi \in \mathcal{M} \} \right\}.$$

Checks. (i) Nonemptiness is immediate because $X \neq \emptyset$ and $\mathcal{M} \neq \emptyset$. (ii) For each $u = \Phi(s)$, the construction enforces $\nu(u) - \nu(s) \in C_\sigma$ (exact for conservative schemes, bounded for dissipative variants). (iii) If $\mathfrak{S}$ is closed under G -conjugation, meaning that for every $g \in G$ and $\Phi \in \mathfrak{S}$ there exists $\Phi' \in \mathfrak{S}$ with $g \circ \Phi = \Phi' \circ g$, then

$$\odot_{\text{step}}^{(1,2)}(g^{\uparrow 1} \cdot X) = g^{\uparrow 2} \cdot \odot_{\text{step}}^{(1,2)}(X).$$

This PSHS represents next-step uncertainty induced by numerical/integrator model choices on ensembles.

Example 34 (Electromagnetic scattering with parameter families: $(m, n) = (1, 2)$). Let S_{phy} be a collection of finite-energy electromagnetic field states $s = (\mathbf{E}, \mathbf{H})$ on a bounded domain Ω with fixed boundary conditions. Let $V = \mathbb{R}_{\geq 0}$ and define the energy valuation

$$\nu(s) = \int_{\Omega} \left(\frac{\varepsilon}{2} \|\mathbf{E}\|^2 + \frac{\mu}{2} \|\mathbf{H}\|^2 \right) dx.$$

Let G be the subgroup of rigid motions preserving Ω (and the boundary data), acting by pull-back; take ρ to be the identity on V .

Consider the unary interaction $\sigma = \text{scatter}$ with uncertain material parameters. Let $\mathfrak{P}$ be a nonempty family of coefficients $p = (\varepsilon, \mu, \sigma_{\text{cond}})$, where $\sigma_{\text{cond}} \geq 0$ models conductive losses. Let $C_\sigma = [0, \Lambda]$ encode that the energy may decrease by at most Λ per event.

For nonempty $X \in \text{PWSET}(S_{\text{phy}})$, define

$$\odot_{\text{scatter}}^{(1,2)}(X) := \left\{ \mathcal{Y} \subseteq S_{\text{phy}} \mid \exists \text{ finite } \mathcal{M} \subseteq \mathfrak{P} \text{ s.t. } \mathcal{Y} = \{ \mathcal{S}_p(s) \mid s \in X, p \in \mathcal{M} \} \right\},$$

where $\mathcal{S}_p$ is the (well-posed) scattering solution operator under parameters p . Then (i) nonemptiness follows from well-posedness; (ii) each atom $u = \mathcal{S}_p(s)$ satisfies $\nu(u) - \nu(s) \in C_\sigma$; and (iii) covariance holds by equivariance of Maxwell's equations under G with the prescribed boundary conditions.

Example 35 (Multiscale coarse-graining from families of ensembles: $(m, n) = (2, 1)$). Let S_{phy} be a set of particle microstates $s = (m, \mathbf{v}, T)$ (mass, velocity, temperature). Let $V = \mathbb{R}^d \times \mathbb{R}$ with

$$\nu(m, \mathbf{v}, T) = \left(m\mathbf{v}, \frac{1}{2}m\|\mathbf{v}\|^2 + c m T \right),$$

and let $G = \text{SE}(d)$ act by rigid motions; ρ is the induced linear action on V . Fix $C_\sigma = \{0\}^d \times [0, \Lambda]$ to model bounded dissipation under coarse-graining.

Inputs now lie in $\text{PWSET}^2((S_{\text{phy}})) = \text{PWSET}(\text{PWSET}(S_{\text{phy}}))$: an element $X \in \text{PWSET}^2((S_{\text{phy}}))$ is a family of ensembles (for example, one ensemble per cell in a spatial partition). Define $\odot_{\text{cg}}^{(2,1)} : \text{PWSET}^2((S_{\text{phy}})) \rightarrow \text{PWSET}(S_{\text{phy}})$ by

$$\begin{aligned} \odot_{\text{cg}}^{(2,1)}(X) := \left\{ (M, \mathbf{u}, \Theta) \mid \exists Y \in X \text{ such that } M = \sum_{s \in Y} m_s, \quad M\mathbf{u} = \sum_{s \in Y} m_s \mathbf{v}_s, \right. \\ \left. \frac{1}{2}M\|\mathbf{u}\|^2 + c M \Theta \geq \sum_{s \in Y} \left(\frac{1}{2}m_s\|\mathbf{v}_s\|^2 + c m_s T_s \right) - \Lambda \right\}. \end{aligned}$$

Checks. (i) If X contains a nonempty ensemble Y , admissible coarse variables exist, hence $\odot_{\text{cg}}^{(2,1)}(X) \neq \emptyset$. (ii) For $u = (M, \mathbf{u}, \Theta) \in \odot_{\text{cg}}^{(2,1)}(X)$ built from Y , every atom $s \in \text{At}(Y)$ satisfies the cone-admissibility relation

$$\nu(u) \in \sum_{s \in Y} \nu(s) + C_\sigma,$$

so momentum is preserved exactly while total energy can drop by at most Λ . (iii) Lifted covariance follows immediately from the geometric nature of the definitions. This $(2, 1)$ -PSHS formalizes coarse-graining from a family of microscopic ensembles to a single macroscopic state.

Proposition 36 (Lifted actions form genuine group actions). *Let G act on S_{phy} . For each $k \geq 0$, the lifted rule*

$$(g, Z) \mapsto g^{\uparrow k} \cdot Z \quad (g \in G, Z \in \text{PWSET}^k((S_{\text{phy}})))$$

defines a left action of G on $\text{PWSET}^k((S_{\text{phy}}))$, i.e.

$$e^{\uparrow k} \cdot Z = Z, \quad (gh)^{\uparrow k} \cdot Z = g^{\uparrow k} \cdot (h^{\uparrow k} \cdot Z) \quad (g, h \in G).$$

Proof: We argue by induction on k .

For $k = 0$, $\text{PWSET}^0((S_{\text{phy}})) = S_{\text{phy}}$ and the statement is exactly the group action on S_{phy} : $e \cdot x = x$ and $(gh) \cdot x = g \cdot (h \cdot x)$.

Assume the claim holds for k , and let $Z \in \text{PWSET}^{k+1}((S_{\text{phy}})) = \text{PWSET}(\text{PWSET}^k((S_{\text{phy}})))$. By definition,

$$e^{\uparrow(k+1)} \cdot Z = \{e^{\uparrow k} \cdot z \mid z \in Z\} = \{z \mid z \in Z\} = Z,$$

where we used the induction hypothesis $e^{\uparrow k} \cdot z = z$.

Similarly,

$$(gh)^{\uparrow(k+1)} \cdot Z = \{(gh)^{\uparrow k} \cdot z \mid z \in Z\} = \{g^{\uparrow k} \cdot (h^{\uparrow k} \cdot z) \mid z \in Z\},$$

by the induction hypothesis on level k . The right-hand side equals

$$\{g^{\uparrow k} \cdot w \mid w \in \{h^{\uparrow k} \cdot z \mid z \in Z\}\} = g^{\uparrow(k+1)} \cdot (h^{\uparrow(k+1)} \cdot Z).$$

Thus the action identities hold at level $k + 1$. $\square$

Lemma 37 (Atoms commute with lifted actions). *For every $k \geq 0$, every $g \in G$, and every $Z \in \text{PWSET}^k((S_{\text{phy}}))$,*

$$\text{At}(g^{\uparrow k} \cdot Z) = g^{\uparrow 0} \cdot \text{At}(Z) = g \cdot \text{At}(Z) \subseteq S_{\text{phy}}.$$

Proof: Induction on k .

For $k = 0$, $Z = x \in S_{\text{phy}}$ and $\text{At}(x) = \{x\}$. Hence

$$\text{At}(g^{\uparrow 0} \cdot x) = \text{At}(g \cdot x) = \{g \cdot x\} = g \cdot \{x\} = g \cdot \text{At}(x).$$

Assume the claim holds for k , and let $Z \in \text{PWSET}^{k+1}((S_{\text{phy}})) = \text{PWSET}(\text{PWSET}^k((S_{\text{phy}})))$. Using the recursive definition of atoms,

$$\text{At}(g^{\uparrow(k+1)} \cdot Z) = \text{At}(\{g^{\uparrow k} \cdot z \mid z \in Z\}) = \bigcup_{z \in Z} \text{At}(g^{\uparrow k} \cdot z).$$

By the induction hypothesis, $\text{At}(g^{\uparrow k} \cdot z) = g \cdot \text{At}(z)$ for each $z \in Z$, hence

$$\text{At}(g^{\uparrow(k+1)} \cdot Z) = \bigcup_{z \in Z} (g \cdot \text{At}(z)) = g \cdot \left(\bigcup_{z \in Z} \text{At}(z) \right) = g \cdot \text{At}(Z).$$

$\square$

Proposition 38 (Equivalent admissibility phrasing). *Let $\sigma \in \Sigma_{\text{phy}}$ with $r = \text{ar}(\sigma)$, and let $\vec{X} = (X_1, \dots, X_r)$. Condition (ii) in Definition 30 is equivalent to the single statement:*

$$\forall u \in \text{At}\left(\odot_{\sigma}^{(m,n)}(\vec{X})\right) \exists s_i \in \text{At}(X_i) \text{ such that } \nu(u) \in \nu(s_1) + \dots + \nu(s_r) + C_{\sigma}.$$

Proof: Set $Z := \odot_{\sigma}^{(m,n)}(\vec{X}) \in \text{PWSET}^n((S_{\text{phy}}))$.

Assume Definition 30(ii). Take any $u \in \text{At}(Z)$. By the recursive rule

$$\text{At}(Z) = \bigcup_{Y \in Z} \text{At}(Y),$$

there exists some $Y \in Z$ with $u \in \text{At}(Y)$. Applying Definition 30(ii) to this Y gives atoms $s_i \in \text{At}(X_i)$ satisfying $\nu(u) \in \nu(s_1) + \dots + \nu(s_r) + C_{\sigma}$.

Conversely, assume the displayed statement holds. Let $Y \in Z$ and $u \in \text{At}(Y)$ be arbitrary. Since $u \in \text{At}(Y) \subseteq \text{At}(Z)$, the displayed statement yields atoms $s_i \in \text{At}(X_i)$ with the same inclusion. Thus Definition 30(ii) holds. $\square$

Theorem 39 (Minkowski form of atomic admissibility). *Let $\sigma \in \Sigma_{\text{phy}}$ have arity r , and let $\vec{X} = (X_1, \dots, X_r)$. Define valuation-images of atomic sets by*

$$\nu(\text{At}(X_i)) := \{ \nu(s) \mid s \in \text{At}(X_i) \} \subseteq V, \quad \nu(\text{At}(Z)) := \{ \nu(u) \mid u \in \text{At}(Z) \} \subseteq V.$$

Then, for $Z = \odot_{\sigma}^{(m,n)}(\vec{X})$,

$$\nu(\text{At}(Z)) \subseteq \nu(\text{At}(X_1)) + \dots + \nu(\text{At}(X_r)) + C_{\sigma}.$$

Proof: Take any $w \in \nu(\text{At}(Z))$. By definition there exists $u \in \text{At}(Z)$ with $w = \nu(u)$. By Proposition 38, there exist atoms $s_i \in \text{At}(X_i)$ such that

$$\nu(u) \in \nu(s_1) + \dots + \nu(s_r) + C_{\sigma}.$$

Hence there exists $c \in C_{\sigma}$ with

$$w = \nu(u) = \nu(s_1) + \dots + \nu(s_r) + c.$$

Since $\nu(s_i) \in \nu(\text{At}(X_i))$, it follows that

$$w \in \nu(\text{At}(X_1)) + \dots + \nu(\text{At}(X_r)) + C_{\sigma}.$$

As w was arbitrary, the inclusion holds. $\square$

Theorem 40 (Covariance transports atomic defect sets). *Let $\sigma \in \Sigma_{\text{phy}}$ have arity r . For $\vec{X} = (X_1, \dots, X_r)$ define the atomic defect set*

$$\Delta_{\sigma}(\vec{X}) := \left\{ \nu(u) - (\nu(s_1) + \dots + \nu(s_r)) \mid u \in \text{At}(\odot_{\sigma}^{(m,n)}(\vec{X})), s_i \in \text{At}(X_i) \right\} \subseteq V.$$

Then for every $g \in G$,

$$\rho(g) \Delta_{\sigma}(\vec{X}) = \Delta_{\sigma}(g^{\uparrow m} \cdot X_1, \dots, g^{\uparrow m} \cdot X_r).$$

Moreover, Definition 30(ii) implies $\Delta_{\sigma}(\vec{X}) \subseteq C_{\sigma}$ for all $\vec{X}$.

Proof: First, $\Delta_{\sigma}(\vec{X}) \subseteq C_{\sigma}$ follows directly from Proposition 38: each defect term equals $\nu(u) - \sum_i \nu(s_i)$ with $\nu(u) \in \sum_i \nu(s_i) + C_{\sigma}$, hence the difference lies in C_{σ} .

Now fix $g \in G$. We show both inclusions.

($\subseteq$) Take any $d \in \Delta_{\sigma}(\vec{X})$. Then

$$d = \nu(u) - \sum_{i=1}^r \nu(s_i)$$

for some $u \in \text{At}(\odot_{\sigma}^{(m,n)}(\vec{X}))$ and $s_i \in \text{At}(X_i)$. By lifted covariance in Definition 30(iii),

$$\odot_{\sigma}^{(m,n)}(g^{\uparrow m} \cdot X_1, \dots, g^{\uparrow m} \cdot X_r) = g^{\uparrow n} \cdot \odot_{\sigma}^{(m,n)}(\vec{X}).$$

Applying Lemma 37 to $Z = \odot_{\sigma}^{(m,n)}(\vec{X})$ yields

$$\text{At}\left(\odot_{\sigma}^{(m,n)}(g^{\uparrow m} \cdot X_1, \dots, g^{\uparrow m} \cdot X_r)\right) = \text{At}(g^{\uparrow n} \cdot Z) = g \cdot \text{At}(Z).$$

Hence $g \cdot u \in \text{At}(\odot_{\sigma}^{(m,n)}(g^{\uparrow m} \cdot X_1, \dots, g^{\uparrow m} \cdot X_r))$. Similarly, Lemma 37 applied to each X_i gives $g \cdot s_i \in \text{At}(g^{\uparrow m} \cdot X_i)$.

Using $\nu(g \cdot x) = \rho(g)\nu(x)$,

$$\rho(g)d = \rho(g)\nu(u) - \sum_{i=1}^r \rho(g)\nu(s_i) = \nu(g \cdot u) - \sum_{i=1}^r \nu(g \cdot s_i) \in \Delta_{\sigma}(g^{\uparrow m} \cdot X_1, \dots, g^{\uparrow m} \cdot X_r).$$

Thus $\rho(g)\Delta_{\sigma}(\vec{X}) \subseteq \Delta_{\sigma}(g^{\uparrow m} \cdot \vec{X})$.

($\supseteq$) Apply the previous inclusion to g^{-1} (using that $\rho(g^{-1}) = \rho(g)^{-1}$), and rewrite:

$$\rho(g^{-1}) \Delta_{\sigma}(g^{\uparrow m} \cdot \vec{X}) \subseteq \Delta_{\sigma}(\vec{X}) \implies \Delta_{\sigma}(g^{\uparrow m} \cdot \vec{X}) \subseteq \rho(g) \Delta_{\sigma}(\vec{X}).$$

Combining both inclusions gives equality. $\square$

Theorem 41 (Atom-level reduction to a Physical HyperStructure). *Assume **PSHS** is a Physical SuperHyperStructure of order (m, n) with $n \geq 1$. Define the singleton-lift embedding $\eta_m : S_{\text{phy}} \rightarrow \text{PWSET}^m(\cdot)_{S_{\text{phy}}}$ recursively by*

$$\eta_0(s) = s, \quad \eta_{k+1}(s) = \{\eta_k(s)\}.$$

For each $\sigma \in \Sigma_{\text{phy}}$ with arity r , define a hyperoperation

$$\sigma^\sharp : S_{\text{phy}}^r \longrightarrow \text{PWSET}(S_{\text{phy}}), \quad \sigma^\sharp(s_1, \dots, s_r) := \text{At}\left(\odot_\sigma^{(m,n)}(\eta_m(s_1), \dots, \eta_m(s_r))\right).$$

Let

$$D_\sigma^\sharp := \left\{ (s_1, \dots, s_r) \in D_\sigma \mid \sigma^\sharp(s_1, \dots, s_r) \neq \emptyset \right\}.$$

Then the tuple

$$\mathbf{PHS}^\sharp = (S_{\text{phy}}, \Sigma_{\text{phy}}, \{\sigma^\sharp\}_{\sigma \in \Sigma_{\text{phy}}}, \nu, G, \{C_\sigma\})$$

is a Physical HyperStructure (Definition 16) with feasibility domains $\{D_\sigma^\sharp\}$.

Proof: Fix $\sigma \in \Sigma_{\text{phy}}$ of arity r .

Feasible nonemptiness on $D_\sigma^\sharp$ is immediate from the definition: if $(s_1, \dots, s_r) \in D_\sigma^\sharp$, then by definition $\sigma^\sharp(s_1, \dots, s_r) \neq \emptyset$.

For admissibility, take any $u \in \sigma^\sharp(s_1, \dots, s_r)$. By definition,

$$u \in \text{At}\left(\odot_\sigma^{(m,n)}(\eta_m(s_1), \dots, \eta_m(s_r))\right).$$

Apply Proposition 38 to $\vec{X} = (\eta_m(s_1), \dots, \eta_m(s_r))$. Then there exist atoms $t_i \in \text{At}(\eta_m(s_i))$ such that

$$\nu(u) \in \nu(t_1) + \dots + \nu(t_r) + C_\sigma.$$

But $\text{At}(\eta_m(s_i)) = \{s_i\}$ by construction of η_m and the recursive definition of atoms, hence $t_i = s_i$. Therefore

$$\nu(u) \in \nu(s_1) + \dots + \nu(s_r) + C_\sigma,$$

which is exactly the admissibility condition for $\sigma^\sharp$.

For covariance, fix $g \in G$. Using Proposition 36 and the definition of η_m , one checks by induction that

$$g^{\uparrow m} \cdot \eta_m(s) = \eta_m(g \cdot s) \quad (s \in S_{\text{phy}}).$$

Then, by lifted covariance of $\odot_\sigma^{(m,n)}$,

$$\odot_\sigma^{(m,n)}(\eta_m(g \cdot s_1), \dots, \eta_m(g \cdot s_r)) = \odot_\sigma^{(m,n)}(g^{\uparrow m} \cdot \eta_m(s_1), \dots, g^{\uparrow m} \cdot \eta_m(s_r)) = g^{\uparrow n} \cdot \odot_\sigma^{(m,n)}(\eta_m(s_1), \dots, \eta_m(s_r)).$$

Applying Lemma 37 to the right-hand side yields

$$\sigma^\sharp(g \cdot s_1, \dots, g \cdot s_r) = \text{At}\left(g^{\uparrow n} \cdot \odot_\sigma^{(m,n)}(\eta_m(s_1), \dots, \eta_m(s_r))\right) = g \cdot \text{At}\left(\odot_\sigma^{(m,n)}(\eta_m(s_1), \dots, \eta_m(s_r))\right) = g \cdot \sigma^\sharp(s_1, \dots, s_r).$$

Thus $\mathbf{PHS}^\sharp$ satisfies the covariance axiom. $\square$

Theorem 42 (Sequential composition of unary physical superhyperoperations). *Let σ be a symbol of arity r with an order- (m, n) superhyperoperation*

$$\odot_\sigma^{(m,n)} : (\text{PWSET}^m(\cdot)_{S_{\text{phy}}})^r \rightarrow \text{PWSET}^n(\cdot)_{S_{\text{phy}}},$$

and let τ be a unary symbol (arity 1) with an order- (n, p) superhyperoperation

$$\odot_\tau^{(n,p)} : \text{PWSET}^n(\cdot)_{S_{\text{phy}}} \rightarrow \text{PWSET}^p(\cdot)_{S_{\text{phy}}}.$$

Assume both satisfy Definition 30 with cones C_σ, C_τ . Define the composite operation of order (m, p) by

$$\odot_{\tau \circ \sigma}^{(m,p)}(\vec{X}) := \odot_\tau^{(n,p)}\left(\odot_\sigma^{(m,n)}(\vec{X})\right), \quad \vec{X} \in (\text{PWSET}^m(\cdot)_{S_{\text{phy}}})^r.$$

Then the atomic admissibility of the composite holds with the summed cone: for every $\vec{X}$ and every $u \in \text{At}(\odot_{\tau \circ \sigma}^{(m,p)}(\vec{X}))$ there exist atoms $s_i \in \text{At}(X_i)$ such that

$$\nu(u) \in \nu(s_1) + \dots + \nu(s_r) + (C_\sigma + C_\tau).$$

Moreover, the lifted covariance of $\odot_{\tau \circ \sigma}^{(m,p)}$ holds automatically.

Proof: Fix $\vec{X} = (X_1, \dots, X_r)$ and set $Z := \odot_{\sigma}^{(m,n)}(\vec{X})$ and $W := \odot_{\tau}^{(n,p)}(Z)$.

Take $u \in \text{At}(W)$. By Proposition 38 applied to the unary operation τ with input Z , there exists an atom $t \in \text{At}(Z)$ such that

$$\nu(u) \in \nu(t) + C_{\tau}.$$

Now apply Proposition 38 to σ and $\vec{X}$: since $t \in \text{At}(Z) = \text{At}(\odot_{\sigma}^{(m,n)}(\vec{X}))$, there exist atoms $s_i \in \text{At}(X_i)$ such that

$$\nu(t) \in \nu(s_1) + \dots + \nu(s_r) + C_{\sigma}.$$

Hence there exist $c_{\tau} \in C_{\tau}$ and $c_{\sigma} \in C_{\sigma}$ with

$$\nu(u) = \nu(t) + c_{\tau} = (\nu(s_1) + \dots + \nu(s_r) + c_{\sigma}) + c_{\tau} = \nu(s_1) + \dots + \nu(s_r) + (c_{\sigma} + c_{\tau}).$$

Since $c_{\sigma} + c_{\tau} \in C_{\sigma} + C_{\tau}$, we obtain

$$\nu(u) \in \nu(s_1) + \dots + \nu(s_r) + (C_{\sigma} + C_{\tau}).$$

For covariance, let $g \in G$. Using Definition 30(iii) for σ ,

$$\odot_{\sigma}^{(m,n)}(g^{\uparrow m} \cdot X_1, \dots, g^{\uparrow m} \cdot X_r) = g^{\uparrow n} \cdot \odot_{\sigma}^{(m,n)}(X_1, \dots, X_r).$$

Apply Definition 30(iii) for τ to the input $Z = \odot_{\sigma}^{(m,n)}(\vec{X})$:

$$\odot_{\tau}^{(n,p)}(g^{\uparrow n} \cdot Z) = g^{\uparrow p} \cdot \odot_{\tau}^{(n,p)}(Z).$$

Combining the two displays gives

$$\odot_{\tau \circ \sigma}^{(m,p)}(g^{\uparrow m} \cdot X_1, \dots, g^{\uparrow m} \cdot X_r) = g^{\uparrow p} \cdot \odot_{\tau \circ \sigma}^{(m,p)}(X_1, \dots, X_r),$$

which is exactly lifted covariance for the composite. $\square$

3 | Conclusion

In this paper, we extended the notions of Hyperstructure and Superhyperstructure to the domain of physics and examined several illustrative applications. In future work, we aim to investigate further extensions of these concepts by incorporating frameworks such as Fuzzy Sets[23], HyperFuzzy Sets[24], Neutrosophic Sets[25], Uncertain Sets[26, 27], Rough Sets[28], and Plithogenic Sets [10, 29]. We also intend to pursue quantitative evaluations and analyses to assess their effectiveness and practical applicability.

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Ethical Approval

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