



Paper Type: Original Article

Hyperfuzzy Petri Nets and n-Superhyperfuzzy Petri Nets: A Generalized Framework for Modeling Uncertainty

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Received: 29 Jan 2025

Revised: 03 Apr 2025

Accepted: 18 Aug 2025

Published: 20 Aug 2025

Abstract

A fuzzy set assigns to each element in a universe a membership degree within the interval $[0, 1]$, effectively capturing imprecision and vagueness. A hyperfuzzy set generalizes this notion by mapping each element to a nonempty subset of $[0, 1]$, thereby representing uncertainty and variability through multiple possible membership degrees. An n -superhyperfuzzy set further extends this framework by assigning to each nonempty element in the n th power-set hierarchy a nonempty subset of $[0, 1]$, thus enabling the modeling of hierarchical and recursively nested uncertainty.

Petri nets are mathematical models designed to describe distributed systems involving conditions, events, and state transitions in concurrent processes. Fuzzy Petri nets enhance Petri nets by integrating fuzzy logic, allowing the representation of uncertainty and partial truth in transitions and states.

In this paper, we define and explore the mathematical structure of Hyperfuzzy Petri Nets and n -SuperHyperfuzzy Petri Nets. These frameworks extend classical Petri nets by incorporating Hyperfuzzy Sets and n -SuperHyperfuzzy Sets, respectively. We hope these generalizations will contribute to the further advancement of research in fuzzy set theory and related fields.

Keywords: Fuzzy petri nets, Fuzzy Set, HyperFuzzy Set, Superhyperfuzzy Set, Petri Nets

1 | Preliminaries

This section provides an overview of the fundamental concepts and definitions essential for the discussions in this paper. In this paper, all sets are assumed to be finite.



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<https://doi.org/10.61356/j.saem.2025.4583>



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1.1 | HyperFuzzy Set and the n -SuperHyperFuzzy Set

A fuzzy set assigns to each element in a universe a membership degree in the interval $[0, 1]$, effectively modeling imprecision and vagueness. The following concepts are known as extensions or related frameworks to fuzzy sets: Intuitionistic Fuzzy Set [1], Vague Set [2], Bipolar Fuzzy Set [3], Neutrosophic Set [4], Neutrosophic Offset[5], Picture Fuzzy Set [6], Hesitant Fuzzy Set [7], Quadri-Partitioned Neutrosophic Set[8, 9], HyperNeutrosophic Set[10], and Plithogenic Set [11].

This paper focuses on Fuzzy Sets and their hierarchical extensions, including the HyperFuzzy Set [12] and the n -SuperHyperFuzzy Set [13]. These advanced frameworks have significant connections to the power set and n -th powerset, which play a crucial role in their definitions and applications.

Definition 1 (Base Set). A *base set* S is the foundational set from which complex structures such as powersets and hyperstructures are derived. It is formally defined as:

$$S = \{x \mid x \text{ is an element within a specified domain}\}.$$

All elements in constructs like $\mathcal{P}(S)$ or $\mathcal{P}_n(S)$ originate from the elements of S .

Definition 2 (Powerset). [14] The *powerset* of a set S , denoted $\mathcal{P}(S)$, is the collection of all possible subsets of S , including both the empty set and S itself. Formally, it is expressed as:

$$\mathcal{P}(S) = \{A \mid A \subseteq S\}.$$

Definition 3 (n -th Powerset). (cf.[15])

The n -th powerset of a set H , denoted $P_n(H)$, is defined iteratively, starting with the standard powerset. The recursive construction is given by:

$$P_1(H) = P(H), \quad P_{n+1}(H) = P(P_n(H)), \quad \text{for } n \geq 1.$$

Similarly, the n -th non-empty powerset, denoted $P_n^*(H)$, is defined recursively as:

$$P_1^*(H) = P^*(H), \quad P_{n+1}^*(H) = P^*(P_n^*(H)).$$

Here, $P^*(H)$ represents the powerset of H with the empty set removed.

Definition 4 (Fuzzy set). [16, 17] A *fuzzy set* τ in a non-empty universe Y is a mapping $\tau : Y \rightarrow [0, 1]$. A *fuzzy relation* on Y is a fuzzy subset δ in $Y \times Y$. If τ is a fuzzy set in Y and δ is a fuzzy relation on Y , then δ is called a *fuzzy relation on τ* if

$$\delta(y, z) \leq \min\{\tau(y), \tau(z)\} \quad \text{for all } y, z \in Y.$$

Definition 5 (Hyperfuzzy Set). [12, 18] Let X be a non-empty universe. A *hyperfuzzy set* $\tilde{A}$ on X is defined by a mapping

$$\tilde{\mu} : X \longrightarrow \tilde{P}([0, 1]),$$

where $\tilde{P}([0, 1])$ denotes the collection of all non-empty subsets of the interval $[0, 1]$.

For each element $x \in X$, $\tilde{\mu}(x) \subseteq [0, 1]$ represents the *set of possible membership degrees* of x in the set $\tilde{A}$. This formulation allows for representing uncertainty or variability in the degree of membership, extending the classical fuzzy set (which assigns a single real number in $[0, 1]$) to a set-valued interpretation.

Thus, a hyperfuzzy set captures both fuzziness and imprecision by associating each element not with a fixed degree, but with a range or subset of plausible membership values.

Definition 6 (n -SuperHyperFuzzy Set). [19] Let X be a non-empty set, and $n \geq 0$ be an integer. An *n -SuperHyperFuzzy Set* is a mapping:

$$\tilde{\mu}_n : \tilde{\mathcal{P}}_n^*(X) \rightarrow \tilde{\mathcal{P}}([0, 1]),$$

where:

- $\tilde{\mathcal{P}}_n^*(X)$ denotes the family of all non-empty elements of the n -th PowerSet $\mathcal{P}_n^*(X)$, defined recursively as:

$$\mathcal{P}_0^*(X) = X, \quad \mathcal{P}_1^*(X) = \mathcal{P}(X), \quad \mathcal{P}_n^*(X) = \mathcal{P}(\mathcal{P}_{n-1}^*(X)), \text{ for } n \geq 2,$$

with $\tilde{\mathcal{P}}_n^*(X) = \mathcal{P}_n^*(X) \setminus \{\emptyset\}$.

- $\tilde{\mathcal{P}}([0, 1])$ denotes the family of all non-empty subsets of the interval $[0, 1]$.

Structure:

- (1) Each element $A \in \tilde{\mathcal{P}}_n^*(X)$ is a non-empty subset within the n -th PowerSet hierarchy of X .
- (2) The mapping $\tilde{\mu}_n$ assigns to each $A \in \tilde{\mathcal{P}}_n^*(X)$ a non-empty subset $\tilde{\mu}_n(A) \subseteq [0, 1]$, representing the degrees of membership associated with the subset A .

Properties:

- If $n = 0$, $\tilde{\mathcal{P}}_0^*(X) = X$, and the structure reduces to a standard fuzzy set:

$$\tilde{\mu}_0 : X \rightarrow [0, 1].$$

- For $n = 1$, $\tilde{\mathcal{P}}_1^*(X) = \tilde{\mathcal{P}}(X)$, and the structure represents a SuperHyperFuzzy Set:

$$\tilde{\mu}_1 : \tilde{\mathcal{P}}(X) \rightarrow \tilde{\mathcal{P}}([0, 1]).$$

- For $n \geq 2$, the structure recursively extends to higher-order fuzzy relationships:

$$\tilde{\mu}_n : \tilde{\mathcal{P}}_n^*(X) \rightarrow \tilde{\mathcal{P}}([0, 1]).$$

The n -SuperHyperFuzzy Set generalizes the concept of fuzzy sets to hierarchical and recursive levels of membership, allowing for higher-order relationships and fuzzy degrees associated with subsets of subsets, and so on, up to the n -th PowerSet hierarchy of X .

1.2 | Fuzzy Petri Nets

Petri nets are mathematical models designed to describe distributed systems involving conditions, events, and state transitions in concurrent processes [20, 21]. A Fuzzy Petri Net models uncertain dynamic systems by combining Petri nets with fuzzy logic for reasoning under imprecision (cf. [22, 23]). A related concept is known as Neutrosophic Petri Nets [24, 25]. The formal definition of Fuzzy Petri Nets is presented below.

Definition 7 (Fuzzy Petri Net). [22] A *fuzzy Petri net* (FPN) is an 8-tuple

$$\mathcal{FPN} = (P, T, M, I, O, w, \theta, CF),$$

where:

- $P = \{p_1, p_2, \dots, p_n\}$ is a finite set of *places*.
- $T = \{t_1, t_2, \dots, t_m\}$ is a finite set of *transitions*.
- $M : P \rightarrow [0, 1]$ is the *marking* (or *fuzzy state*), where $M(p_i) \in [0, 1]$ is the degree to which place p_i holds a token.
- $I : P \times T \rightarrow \{0, 1\}$ is the *input incidence* function, with

$$I(p_i, t_j) = \begin{cases} 1, & \text{if there is an arc } p_i \rightarrow t_j, \\ 0, & \text{otherwise.} \end{cases}$$

- $O : T \times P \rightarrow \{0, 1\}$ is the *output incidence* function, with

$$O(t_j, p_k) = \begin{cases} 1, & \text{if there is an arc } t_j \rightarrow p_k, \\ 0, & \text{otherwise.} \end{cases}$$

- $w : P \times T \cup T \times P \rightarrow [0, 1]$ assigns a *weight* to each arc, indicating the strength of fuzzy influence.

- $\theta : T \rightarrow (0, 1]$ is a *threshold* function: transition t_j is *enabled* when

$$M(p_i) \geq \theta(t_j) \quad \forall p_i \text{ with } I(p_i, t_j) = 1.$$

- $CF : T \times P \rightarrow [0, 1]$ is the *certainty factor*: when t_j fires, the new marking of p_k is updated according to $CF(t_j, p_k)$.

When an enabled transition t_j fires, the marking is updated by

$$M'(p_k) = \max\left(M(p_k), \max_{t_j} \{w(t_j, p_k) \cdot CF(t_j, p_k)\}\right),$$

for each p_k with $O(t_j, p_k) = 1$.

Example 8 (A Simple Fuzzy Inference Net). Consider the FPN

$$\mathcal{FPN} = (P, T, M_0, I, O, w, \theta, CF)$$

with

$$P = \{p_A, p_B, p_C\}, \quad T = \{t_1\},$$

initial marking

$$M_0(p_A) = 0.7, \quad M_0(p_B) = 0.9, \quad M_0(p_C) = 0,$$

incidence functions

$$I(p_A, t_1) = I(p_B, t_1) = 1, \quad O(t_1, p_C) = 1,$$

weights and thresholds

$$w(p_A, t_1) = 0.8, \quad w(p_B, t_1) = 0.6, \quad w(t_1, p_C) = 1.0, \quad \theta(t_1) = 0.5,$$

and certainty factor

$$CF(t_1, p_C) = 0.9.$$

Enabling and firing: Since

$$M_0(p_A) = 0.7 \geq 0.5, \quad M_0(p_B) = 0.9 \geq 0.5,$$

transition t_1 is enabled. Upon firing, the new marking of p_C is

$$M_1(p_C) = \max\{M_0(p_C), w(t_1, p_C) \cdot CF(t_1, p_C)\} = \max\{0, 1.0 \cdot 0.9\} = 0.9.$$

All other places retain their old markings.

Thus this FPN models the fuzzy rule

$$\text{IF "A" with degree 0.7 AND "B" with degree 0.9, THEN "C" with degree 0.9.}$$

2 | Results of This Paper

This section outlines the main results presented in this paper.

2.1 | HyperFuzzy Petri Nets

The definition of the HyperFuzzy Petri Nets is provided below.

Definition 9 (Hyperfuzzy Petri Net). A *hyperfuzzy Petri net* (HFPN) is a tuple

$$\mathcal{HFPN} = (P, T, M, I, O, w, \theta, CF),$$

where:

- $P = \{p_1, \dots, p_n\}$ is a finite set of *places*.
- $T = \{t_1, \dots, t_m\}$ is a finite set of *transitions*.
- $M : P \rightarrow \tilde{\mathcal{P}}([0, 1])$ is the *hyperfuzzy marking*, assigning to each place p_i a nonempty compact subset $M(p_i) \subseteq [0, 1]$.

- $I : P \times T \rightarrow \{0, 1\}$ and $O : T \times P \rightarrow \{0, 1\}$ are the input and output incidence functions as in Definition 7.
- $w : (P \times T) \cup (T \times P) \rightarrow \tilde{\mathcal{P}}([0, 1])$ assigns each arc a hyperfuzzy weight.
- $\theta : T \rightarrow \tilde{\mathcal{P}}([0, 1])$ is a *hyperfuzzy threshold* for enabling.
- $CF : T \times P \rightarrow \tilde{\mathcal{P}}([0, 1])$ is the *hyperfuzzy certainty factor*.

Define the *hyperproduct* of two hyperfuzzy sets $A, B \subseteq [0, 1]$ by

$$A \otimes B = \{a \cdot b \mid a \in A, b \in B\} \subseteq [0, 1].$$

A transition t_j is *enabled* at marking M iff for every p_i with $I(p_i, t_j) = 1$,

$$\sup M(p_i) \geq \sup \theta(t_j).$$

When t_j fires, the new marking $M' : P \rightarrow \tilde{\mathcal{P}}([0, 1])$ is given by

$$M'(p_k) = \begin{cases} M(p_k) \cup (w(t_j, p_k) \otimes CF(t_j, p_k)), & O(t_j, p_k) = 1, \\ M(p_k), & O(t_j, p_k) = 0. \end{cases}$$

Example 10 (Hyperfuzzy Inference Net). Consider the HFPN

$$\mathcal{HFPN} = (P, T, M_0, I, O, w, \theta, CF)$$

with

$$\begin{aligned} P &= \{p_A, p_B, p_C\}, \quad T = \{t_1\}, \\ M_0(p_A) &= \{0.7, 0.75\}, \quad M_0(p_B) = \{0.9\}, \quad M_0(p_C) = \{0\}, \\ I(p_A, t_1) &= I(p_B, t_1) = 1, \quad O(t_1, p_C) = 1, \\ w(p_A, t_1) &= \{0.8\}, \quad w(p_B, t_1) = \{0.6, 0.65\}, \quad w(t_1, p_C) = \{1.0\}, \\ \theta(t_1) &= \{0.5, 0.6\}, \quad CF(t_1, p_C) = \{0.9, 0.95\}. \end{aligned}$$

Enabling:

$$\sup M_0(p_A) = 0.75 \geq \sup \theta(t_1) = 0.6, \quad \sup M_0(p_B) = 0.9 \geq 0.6,$$

so t_1 is enabled.

Firing:

$$w(t_1, p_C) \otimes CF(t_1, p_C) = \{1.0 \cdot 0.9, 1.0 \cdot 0.95\} = \{0.9, 0.95\}.$$

Hence

$$M_1(p_C) = M_0(p_C) \cup \{0.9, 0.95\} = \{0, 0.9, 0.95\}.$$

All other places retain $M_1(p) = M_0(p)$.

Theorem 11. *Every fuzzy Petri net (FPN) can be seen as an HFPN in which all hyperfuzzy sets are singletons. Conversely, if in an HFPN all of $M(p)$, $w(a)$, $\theta(t)$, and $CF(t, p)$ are singletons, then it reduces to the corresponding FPN. Moreover, the marking function $M : P \rightarrow \tilde{\mathcal{P}}([0, 1])$ endows each place with a hyperfuzzy set structure, and similarly for arc weights and certainty factors, so HFPNs carry the full structure of hyperfuzzy sets.*

Proof: (i) $FPN \subseteq HFPN$: Given any FPN $\mathcal{FPN} = (P, T, M, I, O, w, \theta, CF)$, define an HFPN by replacing each scalar membership or weight $\alpha \in [0, 1]$ by the singleton set $\{\alpha\} \subseteq [0, 1]$. All definitions of enabling and firing coincide, since singleton hyperproducts $\{\alpha\} \otimes \{\beta\} = \{\alpha\beta\}$ reproduce the FPN update.

(ii) $HFPN$ with singletons $\rightarrow FPN$: If in an HFPN every hyperfuzzy set is a singleton, then $M(p) = \{\mu_p\}$, etc. One recovers the original FPN by identifying each singleton $\{\mu_p\}$ with the scalar μ_p . The enabling condition and firing rule reduce to those of Definition 7.

Hyperfuzzy structure: By construction, M maps places into $\tilde{\mathcal{P}}([0, 1])$, the collection of all nonempty subsets of $[0, 1]$. Thus each place carries a hyperfuzzy membership set. Likewise, w and CF take values in $\tilde{\mathcal{P}}([0, 1])$ and θ in $\tilde{\mathcal{P}}([0, 1])$, exhibiting the hyperfuzzy set structure on arcs and thresholds. Therefore HFPNs naturally generalize FPNs and realize the full power of hyperfuzzy sets. $\square$

Theorem 12 (Monotonicity of Markings). *Let $\mathcal{HFPN} = (P, T, M, I, O, w, \theta, CF)$ be a hyperfuzzy Petri net, and let $t_j \in T$ be enabled at marking M . Denote by M' the marking after t_j fires. Then for every place $p \in P$,*

$$\sup M'(p) \geq \sup M(p).$$

In particular, firing a transition never decreases the maximum membership degree of any place.

Proof: By definition, if $O(t_j, p) = 1$ then

$$M'(p) = M(p) \cup (w(t_j, p) \otimes CF(t_j, p)),$$

while if $O(t_j, p) = 0$ then $M'(p) = M(p)$. In either case,

$$\sup M'(p) = \sup(M(p) \cup (w(t_j, p) \otimes CF(t_j, p))) = \max\{\sup M(p), \sup(w(t_j, p) \otimes CF(t_j, p))\} \geq \sup M(p).$$

If $O(t_j, p) = 0$ the equality is trivial. Hence the claim. $\square$

Theorem 13 (Idempotency of Repeated Firings). *In the same setting as Theorem 12, suppose transition t_j fires twice in succession (with no other firings in between). Then the second firing has no additional effect on any place marking; that is, if M'' is the marking after the second firing, then*

$$M''(p) = M'(p) \quad \text{for all } p \in P.$$

Proof: After the first firing, for each p with $O(t_j, p) = 1$,

$$M'(p) = M(p) \cup (w(t_j, p) \otimes CF(t_j, p)).$$

In order for t_j to remain enabled a second time, we must have $\sup M'(q) \geq \sup \theta(t_j)$ for every input place q . But then the second firing updates

$$M''(p) = M'(p) \cup (w(t_j, p) \otimes CF(t_j, p)).$$

Since $(w \otimes CF) \subseteq M'(p)$ already holds after the first firing (those hyperproduct values were adjoined in M'), we get

$$M''(p) = M'(p) \cup M'(p) = M'(p).$$

For places with $O(t_j, p) = 0$, trivially $M''(p) = M'(p)$. This completes the proof. $\square$

Theorem 14 (Commutativity of Independent Transitions). *Let $t_j, t_k \in T$ be two transitions in a hyperfuzzy Petri net such that they have disjoint sets of input places: $\{p \mid I(p, t_j) = 1\} \cap \{p \mid I(p, t_k) = 1\} = \emptyset$. Then, whenever both are enabled at some marking M , firing t_j and then t_k yields the same final marking as firing t_k and then t_j .*

Proof: Let M_j be the marking after firing t_j from M , and M_{jk} the marking after subsequently firing t_k . Because t_k 's input places are disjoint from those of t_j , the enabling condition and the hyperproduct updates for t_k see the same $\sup M(p)$ whether we use M or M_j . Hence t_k is still enabled at M_j , and

$$M_{jk}(p) = M_j(p) \cup (w(t_k, p) \otimes CF(t_k, p)).$$

But $M_j(p)$ itself equals

$$M(p) \cup (w(t_j, p) \otimes CF(t_j, p)),$$

and for any place p , the two unions commute and adjoin disjoint hyperproduct values. One checks similarly that firing t_k first and then t_j yields

$$M_{kj}(p) = M(p) \cup (w(t_k, p) \otimes CF(t_k, p)) \cup (w(t_j, p) \otimes CF(t_j, p)),$$

which plainly equals $M_{jk}(p)$. Thus the two firings commute. $\square$

2.2 | n-SuperHyperfuzzy Petri Nets

The definition of the n -SuperHyperfuzzy Petri Nets is provided below.

Definition 15 (n -SuperHyperfuzzy Petri Net). Let $n \geq 1$ be an integer. An n -superhyperfuzzy Petri net (n-SHFPN) is a tuple

$$\mathcal{SHFPN}_n = (P, T, M, I, O, w, \theta, CF),$$

where

- $P = \{p_1, \dots, p_N\}$ is a finite set of *places*.
- $T = \{t_1, \dots, t_M\}$ is a finite set of *transitions*.
- $M : P \rightarrow \tilde{\mathcal{P}}_n^*([0, 1])$ is the n -superhyperfuzzy marking: each place p is assigned a nonempty element of the n th nonempty powerset of $[0, 1]$.
- $I : P \times T \rightarrow \{0, 1\}$ and $O : T \times P \rightarrow \{0, 1\}$ are the usual input/output incidence functions.
- $w : (P \times T) \cup (T \times P) \rightarrow \tilde{\mathcal{P}}_n^*([0, 1])$ assigns each arc a hyperfuzzy weight of order n .
- $\theta : T \rightarrow \tilde{\mathcal{P}}_n^*((0, 1])$ is the n -superhyperfuzzy threshold.
- $CF : T \times P \rightarrow \tilde{\mathcal{P}}_n^*([0, 1])$ is the n -superhyperfuzzy certainty factor.

Define recursively the n -superhyperproduct

$$A \otimes_n B = \{X \otimes_{n-1} Y \mid X \in A, Y \in B\},$$

with $\otimes_1$ the usual hyperproduct $X \otimes_1 Y = \{x \cdot y \mid x \in X, y \in Y\}$.

A transition t_j is *enabled* at marking M if

$$\sup^{(n)} M(p_i) \geq \sup^{(n)} \theta(t_j) \quad \text{for all } p_i \text{ with } I(p_i, t_j) = 1,$$

where $\sup^{(n)}$ denotes the n -fold supremum in $\tilde{\mathcal{P}}_n^*$. When t_j fires, the new marking M' is

$$M'(p_k) = \begin{cases} M(p_k) \cup (w(t_j, p_k) \otimes_n CF(t_j, p_k)), & O(t_j, p_k) = 1, \\ M(p_k), & O(t_j, p_k) = 0. \end{cases}$$

Example 16 (A 2-SuperHyperfuzzy Inference Net). Let

$$P = \{p_A, p_B, p_C\}, \quad T = \{t_1\},$$

and define a 2-SHFPN by

$$\begin{aligned} M_0(p_A) &= \{\{0.7, 0.75\}, \{0.8\}\}, \quad M_0(p_B) = \{\{0.9\}\}, \quad M_0(p_C) = \{\{0\}\}, \\ I(p_A, t_1) &= I(p_B, t_1) = 1, \quad O(t_1, p_C) = 1, \\ w(p_A, t_1) &= \{\{0.8\}\}, \quad w(p_B, t_1) = \{\{0.6, 0.65\}\}, \quad w(t_1, p_C) = \{\{1.0\}\}, \\ \theta(t_1) &= \{\{0.5\}\}, \quad CF(t_1, p_C) = \{\{0.9, 0.95\}\}. \end{aligned}$$

Enabling: $\sup^{(2)} M_0(p_A) = 0.8 \geq 0.5 = \sup^{(2)} \theta(t_1)$ and $\sup^{(2)} M_0(p_B) = 0.9 \geq 0.5$, so t_1 is enabled.

Firing:

$$w(t_1, p_C) \otimes_2 CF(t_1, p_C) = \{\{0.8\} \otimes_1 \{0.9, 0.95\}\} = \{\{0.72, 0.76\}\}.$$

Hence

$$M_1(p_C) = M_0(p_C) \cup \{\{0.72, 0.76\}\} = \{\{0\}, \{0.72, 0.76\}\}.$$

Other places remain at $M_1 = M_0$.

Example 17 (A 3-SuperHyperfuzzy Inference Net). Let $P = \{p_A, p_B, p_C\}$ and $T = \{t_1\}$. We define a 3-superhyperfuzzy Petri net $\mathcal{SHFPN}_3 = (P, T, M_0, I, O, w, \theta, CF)$ by:

$$\begin{aligned} M_0(p_A) &= \{\{\{0.6, 0.65\}, \{0.7\}\}, \{\{0.8, 0.85\}\}\} \in \tilde{\mathcal{P}}_3^*([0, 1]), \\ M_0(p_B) &= \{\{\{0.9\}\}\}, \\ M_0(p_C) &= \{\{\{0\}\}\}. \end{aligned}$$

The incidence functions are

$$I(p_A, t_1) = I(p_B, t_1) = 1, \quad O(t_1, p_C) = 1,$$

and all other entries are zero. The arc-weight, threshold, and certainty-factor mappings are:

$$\begin{aligned} w(p_A, t_1) &= \{\{\{0.5, 0.55\}\}\}, & w(p_B, t_1) &= \{\{\{0.6\}, \{0.65\}\}\}, & w(t_1, p_C) &= \{\{\{1.0\}\}\}, \\ \theta(t_1) &= \{\{\{0.8\}\}, \{\{0.75, 0.78\}\}\}, & CF(t_1, p_C) &= \{\{\{0.9\}\}, \{\{0.95\}\}\}. \end{aligned}$$

Enabling condition. Compute the 3-fold suprema:

$$\sup^{(3)} M_0(p_A) = \max\{\max\{0.7, 0.65\}, 0.85\} = 0.85, \quad \sup^{(3)} M_0(p_B) = 0.9.$$

Also

$$\sup^{(3)} \theta(t_1) = \max\{0.8, 0.78\} = 0.8.$$

Since $0.85 \geq 0.8$ and $0.9 \geq 0.8$, transition t_1 is enabled.

Firing and update. The 3-superhyperproduct $\{\{\{1.0\}\}\} \otimes_3 \{\{\{0.9\}\}\}$ yields $\{\{\{0.9\}\}\}$, and similarly $\{\{\{1.0\}\}\} \otimes_3 \{\{\{0.95\}\}\} = \{\{\{0.95\}\}\}$. Thus

$$w(t_1, p_C) \otimes_3 CF(t_1, p_C) = \{\{\{0.9\}\}, \{\{0.95\}\}\}.$$

Updating the marking of p_C gives

$$M_1(p_C) = M_0(p_C) \cup \{\{\{0.9\}\}, \{\{0.95\}\}\} = \{\{\{0\}\}, \{\{0.9\}\}, \{\{0.95\}\}\}.$$

All other places retain their original 3-superhyperfuzzy marks.

Hence $\mathcal{SHFPN}_3$ models a hierarchical fuzzy inference:

$$\text{IF "A" in } \{0.6, 0.65, 0.7, 0.8, 0.85\} \text{ AND "B" in } \{0.9\}, \text{ THEN "C" in } \{0.9, 0.95\},$$

at three nested levels of uncertainty.

Example 18 (Smart Thermostat Control as a 4-SuperHyperfuzzy Petri Net). Consider a smart home thermostat whose goal is to maintain a comfortable temperature despite uncertain sensor readings and user preferences. Model this as a 4-superhyperfuzzy Petri net $\mathcal{SHFPN}_4 = (P, T, M_0, I, O, w, \theta, CF)$ with

$$P = \{p_{temp}, p_{set}, p_{action}\}, \quad T = \{t_{ctrl}\}.$$

Initial 4-SHF Markings.

$$\begin{aligned} M_0(p_{temp}) &= \left\{ \underbrace{\{\{\{0.20, 0.25\}\}, \{\{0.30\}\}\}}_A, \underbrace{\{\{\{0.15, 0.35\}\}\}}_B \right\}, \\ M_0(p_{set}) &= \{\{\{\{0.22\}\}\}\}, \\ M_0(p_{action}) &= \{\{\{\{0\}\}\}\}. \end{aligned}$$

Here each entry is a nonempty element of $\tilde{\mathcal{P}}_4^*([0, 1])$; for instance,

$$A = \left\{ \underbrace{\{\{\{0.20, 0.25\}\}}_{\text{sensor cluster I}}, \underbrace{\{\{0.30\}\}}_{\text{cluster II}} \right\}.$$

Incidence Functions.

$$I(p_{temp}, t_{ctrl}) = I(p_{set}, t_{ctrl}) = 1, \quad O(t_{ctrl}, p_{action}) = 1.$$

Arc Weights, Threshold, and Certainty Factor.

$$\begin{aligned}
w(p_{temp}, t_{ctrl}) &= \{\{\{\{0.8, 0.85\}\}, \{\{0.9\}\}\}, \\
w(p_{set}, t_{ctrl}) &= \{\{\{\{0.95\}\}\}\}, \\
w(t_{ctrl}, p_{action}) &= \{\{\{\{1.0\}\}\}\}, \\
\theta(t_{ctrl}) &= \{\{\{\{0.7\}\}\}\}, \\
CF(t_{ctrl}, p_{action}) &= \{\{\{\{0.9\}\}\}\}.
\end{aligned}$$

Enabling. Compute the 4-fold suprema:

$$\sup^{(4)} M_0(p_{temp}) = \max\{\max\{0.25, 0.30\}, 0.35\} = 0.35, \quad \sup^{(4)} M_0(p_{set}) = 0.22,$$

and

$$\sup^{(4)} \theta(t_{ctrl}) = 0.7.$$

Since $0.35 < 0.7$, the transition t_{ctrl} is *not* yet enabled (temperature reading is too low relative to setpoint demands).

Firing after Temperature Rises. Suppose a new temperature measurement arrives, expanding $M_0(p_{temp})$ by firing a hypothetical measurement transition (not shown) to include a 4-SHF value $\{\{\{\{0.8\}\}\}\}$. Then

$$M_1(p_{temp}) = M_0(p_{temp}) \cup \{\{\{\{0.8\}\}\}\}, \quad \sup^{(4)} M_1(p_{temp}) = 0.8 \geq 0.7,$$

so now t_{ctrl} is enabled.

Firing t_{ctrl} . The 4-superhyperproduct is

$$w(t_{ctrl}, p_{action}) \otimes_4 CF(t_{ctrl}, p_{action}) = \{\{\{\{1.0 \cdot 0.9\}\}\}\} = \{\{\{\{0.9\}\}\}\}.$$

Updating gives

$$M_2(p_{action}) = M_1(p_{action}) \cup \{\{\{\{0.9\}\}\}\} = \{\{\{\{0\}\}\}, \{\{\{\{0.9\}\}\}\}.$$

Thus this 4-SHFNP models hierarchical uncertainty in sensor fusion and user preferences, and it triggers the appropriate heating/cooling action only when the multi-level fuzzy conditions are met.

Theorem 19. *Every hyperfuzzy Petri net can be regarded as a 1-superhyperfuzzy Petri net, and conversely any n-superhyperfuzzy Petri net whose data all lie in $\tilde{\mathcal{P}}_1^*([0, 1])$ reduces to a hyperfuzzy Petri net. Hence n-SHFNPns strictly generalize both FPNs and HFPNs.*

Proof: (a) **HFPN $\implies$ 1-SHFNP.** Let $\mathcal{HFPN} = (P, T, M, I, O, w, \theta, CF)$ be a hyperfuzzy Petri net, so

$$M, w, CF : \cdot \longrightarrow \tilde{\mathcal{P}}([0, 1]), \quad \theta : T \rightarrow \tilde{\mathcal{P}}((0, 1]).$$

But by definition $\tilde{\mathcal{P}}([0, 1]) = \tilde{\mathcal{P}}_1^*([0, 1])$ and $\tilde{\mathcal{P}}((0, 1]) = \tilde{\mathcal{P}}_1^*((0, 1])$. Hence all marking, weight, threshold and certainty-factor maps already take values in the 1st level of the nonempty powerset hierarchy. All enabling and firing rules for an HFPN coincide verbatim with those of a 1-SHFNP.

(b) **1-SHFNP $\implies$ HFPN.** Conversely, let $\mathcal{SHFNP}_n$ be an n-superhyperfuzzy net and assume

$$M, w, CF, \theta : \cdot \longrightarrow \tilde{\mathcal{P}}_1^*([0, 1]) \text{ (or } \tilde{\mathcal{P}}_1^*((0, 1]) \text{ for } \theta).$$

Since $\tilde{\mathcal{P}}_1^*([0, 1]) = \tilde{\mathcal{P}}([0, 1])$, each map exactly matches the codomain of a hyperfuzzy Petri net. The definitions of enabling (via sup) and firing (via the hyperproduct $\otimes = \otimes_1$) therefore reproduce the HFPN semantics.

(c) **Proper Generalization.** By construction, an n-SHFNP permits each of M, w, θ, CF to range over $\tilde{\mathcal{P}}_n^*([0, 1])$, a strictly larger family for $n > 1$. Thus every FPN or HFPN (where all values lie in singletons or 1-level sets) embeds into an n-SHFNP, but in general an n-SHFNP cannot be collapsed to a hyperfuzzy or ordinary fuzzy net when $n > 1$. This completes the proof that n-superhyperfuzzy Petri nets both encompass and properly extend the fuzzy and hyperfuzzy models. $\square$

Theorem 20 (Monotonicity of n -Markings). *Let $\mathcal{SHFPN}_n = (P, T, M, I, O, w, \theta, CF)$ be an n -superhyperfuzzy Petri net, and let $t_j \in T$ be enabled at marking M . Denote by M' the marking after t_j fires. Then for every place $p \in P$,*

$$\sup^{(n)} M'(p) \geq \sup^{(n)} M(p),$$

i.e. firing a transition never decreases the n -fold supremum of any place's marking.

Proof: If $O(t_j, p) = 1$, then by definition

$$M'(p) = M(p) \cup (w(t_j, p) \otimes_n CF(t_j, p)),$$

while if $O(t_j, p) = 0$, $M'(p) = M(p)$. In the former case,

$$\sup^{(n)} M'(p) = \sup^{(n)} (M(p) \cup (w \otimes_n CF)) = \max\{\sup^{(n)} M(p), \sup^{(n)} (w \otimes_n CF)\} \geq \sup^{(n)} M(p),$$

and in the latter case the equality is immediate. This proves the claim. $\square$

Theorem 21 (Idempotency of Repeated Firings). *Under the same assumptions as Theorem 20, suppose transition t_j fires twice in succession (with no intervening firings). Let M'' be the marking after the second firing. Then*

$$M''(p) = M'(p) \quad \text{for all } p \in P.$$

Proof: After the first firing,

$$M'(p) = M(p) \cup (w(t_j, p) \otimes_n CF(t_j, p)).$$

Since $\sup^{(n)} M'(q) \geq \sup^{(n)} \theta(t_j)$ holds for each input place q , t_j remains enabled. Hence the second firing gives

$$M''(p) = M'(p) \cup (w(t_j, p) \otimes_n CF(t_j, p)) = M'(p),$$

because the set $w \otimes_n CF$ was already adjoined in $M'(p)$. If $O(t_j, p) = 0$ then $M''(p) = M'(p)$ trivially. This completes the proof. $\square$

3 | Conclusion and Future Works

In this paper, we defined and investigated the mathematical structures of *Hyperfuzzy Petri Nets* and *n-SuperHyperfuzzy Petri Nets*. These frameworks extend classical Petri nets by incorporating Hyperfuzzy Sets and n -SuperHyperfuzzy Sets, respectively, enabling more expressive modeling of uncertainty and hierarchical dynamics.

As future work, we aim to explore further extensions using *hypergraphs* [26, 27] and *superhypergraphs* [28, 29, 14]. In addition, we hope to examine potential real-world applications and conduct computational experiments to validate and implement these theoretical models.

Funding

This research received no external funding.

Data Availability

This paper does not involve any data analysis.

Ethical Approval

This article does not involve any research with human participants or animals.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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