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Two Applications of the Concepts of Pole and Polar with Respect to a Circle

Ion Patrascu ^{1,*}  and Florentin Smarandache ² ¹ Buzesti Brothers National College, Craiova, Romania; patrascu@ipad.ro.² Department of Mathematics & Sciences, University of New Mexico, Gallup, NM 87301, USA; smarand@unm.edu.**Received:** 22 Sep 2023**Revised:** 14 Dec 2023**Accepted:** 16 Jan 2024**Published:** 19 Jan 2024

Abstract

In this article, we present in a unified form the theoretical results regarding the concepts of pole and polar with respect to a circle, and as applications of these, we offer a demonstration of Newton's theorem related to the circumscribed quadrilateral and a demonstration of a theorem usually derived as a particular case of Pascal's theorem related to the inscribed hexagon.

Keywords: Pole, Geometric Transformations, Inversion, Projective Geometry, Duality, Orthogonality, Conjugate Diameters, Conjugate Points, Circle Geometry, Polar.

1 | Introduction

In the realm of geometry, few concepts are as elegant and versatile as the notions of pole and polar with respect to a circle. These concepts, rooted in projective geometry, have found profound applications across various mathematical disciplines and beyond. In this paper, we explore two fascinating applications that highlight the beauty and utility of these concepts.

The concept of pole and polar is deeply intertwined with the idea of duality in projective geometry. Duality, a fundamental principle, establishes a correspondence between points and lines in a plane, wherein statements about points translate to statements about lines and vice versa. This duality underpins the power of the pole-polar relationship, allowing us to seamlessly transition between geometric entities [1].

Our exploration begins with a discussion on the basic definitions of pole and polar with respect to a circle. We then delve into our first application, which demonstrates how these concepts provide a geometric framework for understanding the orthogonality of circles. This application not only reveals the elegance of pole and polar but also showcases their utility in solving practical problems, such as determining the common tangents to two circles [2].

Moving forward, we shift our focus to the second application, which explores the use of poles and polars in constructing inversive transformations. Inversion, a powerful tool in geometry and complex analysis, can be elegantly described and analyzed using the concepts of pole and polar. Through this application, we highlight the deep connections between seemingly disparate areas of mathematics [3].

**Corresponding Author:** smarand@unm.edu<https://doi.org/10.61356/j.scin.2024.11001>

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This paper aims to illuminate the beauty and versatility of the concepts of pole and polar with respect to a circle. By showcasing their applications in the context of orthogonality of circles and inversive transformations, we hope to inspire further exploration and appreciation of these fundamental ideas.

2 | Theoretical Concepts Regarding the Notions of Pole and Polar with Respect to a Circle

Definition 1: Let $\Gamma(O, r)$ be a given circle, and P an external point to the circle. If a secant through P intersects the circle at points M and N , then the point Q that lies on the chord MN and satisfies the condition $\frac{PM}{PN} = \frac{QM}{QN}$ (1) is said to be the harmonic conjugate of point P with respect to points M and N or that it is the harmonic conjugate of point P with respect to the circle.

It can also be said that point Q is the harmonic conjugate of P with respect to the circle if condition (1) is met.

Definition 2: If P is a point in the plane of the circle $\Gamma(O, r)$, $P \neq O$ and Q is its conjugate with respect to the circle, then the line perpendicular to OP drawn from Q is said to be the polar of point P with respect to the circle [4-5].

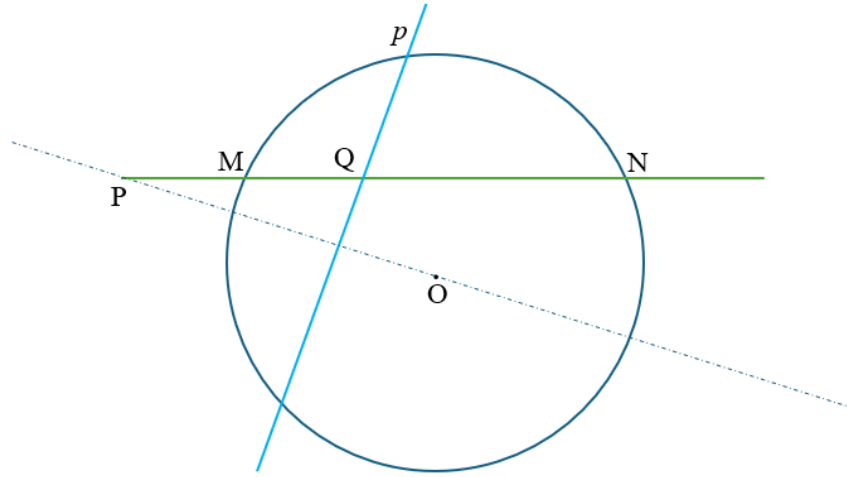


Figure 1. Polar of point P with respect to circle $\Gamma(O, r)$.

Observation 1: In Figure 1, the line p is the polar of P with respect to the circle.

Remark 1:

- a) If $P \in \Gamma(O, r)$, the polar of P is the tangent drawn at P to the circle.
- b) If P is an interior point of the circle, its polar is a line outside the circle.

Proposition 1: If p is the polar of point P outside the circle $\Gamma(O, r)$ with respect to it, and P' is the intersection of the polar with the line OP , then: $OP \cdot OP' = r^2$.

Proof

Let Q be the harmonic conjugate of P with respect to the circle, then: $QP' \perp OP$ and $\frac{PM}{PN} = \frac{QM}{QN}$ (1) – see Figure 2.

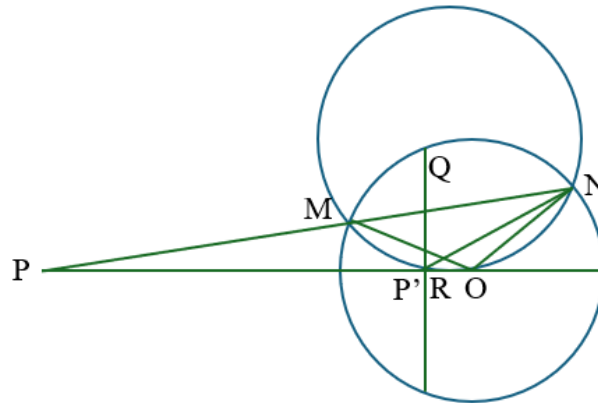


Figure 2. Illustration for Proposition 1. Point P' is the intersection of the polar p with line OP .

If R is the point in which the circumscribed circle of the triangle MON re-intersects the line OP , because $\sphericalangle OMP = \sphericalangle ORM$ (have the same supplement), $\sphericalangle OMN = \sphericalangle ONM$ and $\sphericalangle MOP = \sphericalangle ROM$, it follows that $\triangle OMP \sim \triangle ORM$.

From here, we note that $\frac{OM}{OR} = \frac{OP}{OM}$, meaning that $OP \cdot OR = r^2$ (2).

From the inscriptible quadrilateral $NORM$ we have that $\sphericalangle MRP = \sphericalangle ONM$ (3).

But $ON = OM$, therefore $\sphericalangle ONM = \sphericalangle OMN$ (4).

We also have $\sphericalangle NRO = \sphericalangle OMN$ (5).

The relations (3), (4) and (5) lead to $\sphericalangle MRP = \sphericalangle NRO$, which shows that RP is the external angle bisector $\sphericalangle MRN$.

If we denote by Q' the foot of the internal angle bisector $\sphericalangle MRN$, we obtain that $\frac{Q'M}{Q'N} = \frac{RM}{RN}$.

But $\frac{PM}{PN} = \frac{RM}{RN}$ (angle bisector theorem), we also have $\frac{PM}{PN} = \frac{Q'M}{Q'N}$ (6).

The relations (6) and (1) show that $Q' = Q$, therefore $QR \perp RP$.

But since $QP' \perp OP$, from the uniqueness of the perpendicular drawn from a point to a line, it follows that points R and P' coincide. Then relation (2) becomes $OP \cdot OP' = r^2$, which is what needed to be proven.

Remark 2:

- From the proof of this property, it follows that the harmonic conjugate points of P with respect to the circle $\Gamma(O, r)$ belong to the polar of point P .
- If PU and PV are the tangents drawn from P to the circle $\Gamma(O, r)$ and $UV \cap OP = \{P'\}$, then from the cathetus theorem applied in the right triangles POU and POV , we have $OU^2 = OV^2 = r^2 = OP \cdot OP'$, so points U and V belong to the polar of P with respect to the circle.

This remark justifies the construction of the polar p of P with respect to circle $\Gamma(O, r)$ as the line determined by the points of contact of the tangents drawn from P to the circle [6].

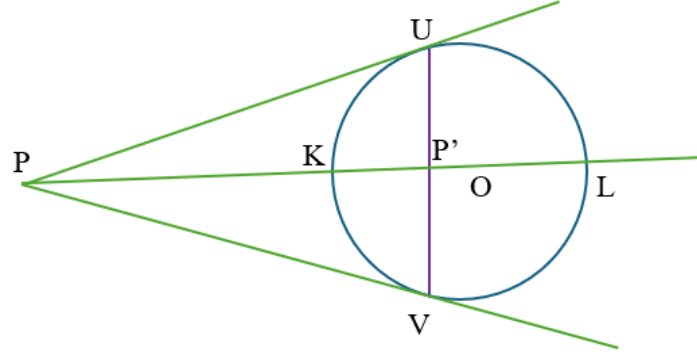


Figure 3. Illustration showing the point P' as the harmonic conjugate of P with respect to the circle $\Gamma(O, r)$.

- c) The point P' in Figure 3 is the harmonic conjugate of P with respect to the circle.

Indeed, if we denote by K and L the intersection of the secant PO with the circle Γ and $OP = d$, $OP' = x$, therefore $PL = d + r$, $PK = d - r$, $P'K = r - x$.

We know that $OP \cdot OP' = r^2$, therefore $d \cdot x = r^2$.

$$\frac{PK}{PL} = \frac{d-r}{d+r}, \frac{P'K}{P'L} = \frac{r-x}{r+x}.$$

$$\frac{PK}{PL} = \frac{P'K}{P'L} \Leftrightarrow \frac{d-r}{d+r} = \frac{r-x}{r+x} \Leftrightarrow d \cdot x = r^2.$$

- d) From the demonstrated results, it follows that the polar of a point contains the harmonic conjugates of that point with respect to the circle.

Theorem 1 (Characterization of the Polar)

The point M belongs to the polar of point P with respect to circle $\Gamma(O, r)$ if and only if $MO^2 - MP^2 = 2r^2 - OP^2$ (1).

Proof

We denote by p the polar of P with respect to Γ (see Figure 4) and let P' be the projection of M on OP . Then $MO^2 - MP^2 = P'O^2 - P'P^2 = OU^2 - UP^2 = r^2 - (OP^2 - r^2) = 2r^2 - OP^2$. U is the point of contact between $\Gamma(O, r)$ and the tangent drawn from P .

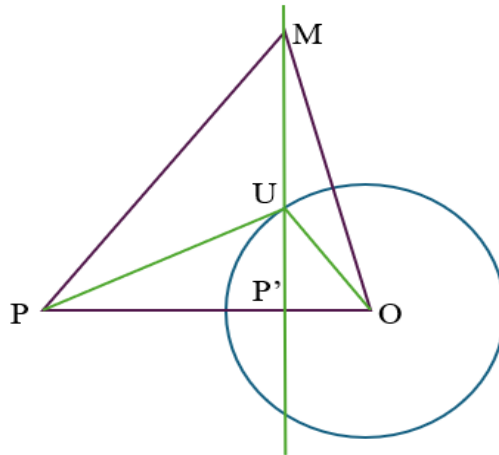


Figure 4. Illustration showing the polar p of point P with respect to circle Γ .

Reciprocal: If M is a point in the plane of the circle $\Gamma(O, r)$ and it satisfies the relation (1), denoting by $M' = pr_{OP}^M$, we have: $M'O^2 - M'P^2 = (MO^2 - M'M^2) - (MP^2 - M'M^2) = MO^2 - MP^2 = 2r^2 - OP^2$ (2).

On the other hand, $P'O^2 - P'P^2 = 2r^2 - OP^2$ (3).

The relations (2) and (3) show that $M' = P'$, therefore M belongs to the polar p of P with respect to Γ .

Theorem 2 (Philippe de La Hire)

If P, Q, R are points in the plane of circle $\Gamma(O, r)$, and p, q, r are their respective polars, then:

1. $P \in q \Leftrightarrow Q \in p$ (If a point belongs to the polar of another point, then this second point belongs to the polar of the first point).
2. If $R \in p \cap q \Leftrightarrow PQ = r$ (The pole of a line determined by two points is the intersection of the polars of these points with respect to a given circle).

Proof:

From *Theorem 1*, if $P \in q \Leftrightarrow PO^2 - PQ^2 = 2r^2 - OQ^2$. This relationship is equivalent to $QO^2 - PO^2 = 2r^2 - OP^2 \Leftrightarrow Q \in P$.

From $R \in p \cap q$ and (1), it follows that $P \in r$ and $Q \in r$, therefore $r = PQ$.

Remark 3: From this theorem, we note:

- a) The polar of a point, which is the intersection of two lines, is determined by the poles of these lines with respect to a given circle.
- b) The poles of concurrent lines are collinear points, and conversely: the polars of collinear points are concurrent lines.

Theorem 3: Let $ABCD$ be a convex quadrilateral inscribed in the circle Γ . Let P be the intersection of the lines AB and AC , Q the intersection of the lines BC and AD , and $\{R\} = AC \cap BD$. Let U and V be the points of contact between Γ and the tangents drawn from P , and K and L the intersections of the tangents drawn at A and B , respectively at C and D , with Γ .

Then the points Q, K, U, R, V and L belong to the polar of point P with respect to the circle.

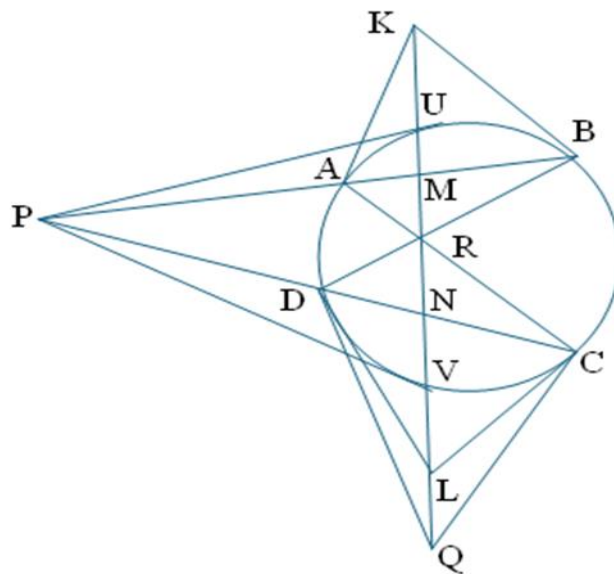


Figure 5. Illustration of the quadrilateral $ABCD$ inscribed in circle Γ .

Proof

We denote by $\{M\} = QR \cap AB$ and $\{N\} = QR \cap CD$ (see *Figure 5*). We prove that M is the conjugate of P with respect to A and B and that N is the conjugate of P with respect to D and C .

It is sufficient to show that $\frac{PA}{PB} = \frac{MA}{MB}$. We apply Ceva's theorem and Menelaus' theorem in triangle QAB for the transversal $P - D - C$. We have:

$$\frac{MA}{MB} \cdot \frac{CB}{QC} \cdot \frac{DQ}{DA} = 1, (1)$$

$$\frac{PA}{PB} \cdot \frac{BC}{QC} \cdot \frac{DQ}{DA} = 1. (2)$$

From the relations (1) and (2) we obtain that $\frac{PA}{PB} = \frac{MA}{MB}$.

Similarly, it can be shown that $\frac{PD}{PC} = \frac{ND}{NC}$.

The points M and N belong to the polar of the point P , so this is the line QR .

From Theorem 2, we know that the polar of point $\{P\} = AB \cap CD$ is the line determined by the poles of lines AB and CD , namely K and L . Therefore, points K and L are also on the polar of P with respect to circle Γ . On the other hand, we have seen that the polar of P is determined by the points of contact of the tangents drawn from P to Γ , which are points U and V . Since the polar of a point with respect to a circle is a unique line, it follows that the aforementioned points are collinear.

Observation 2:

From Theorem 2, it follows that the polar of point Q is line PR , and the polar of point R is line PQ .

3 | Applications of the Concepts of Pole and Polar with Respect to a Circle

Theorem 4 (I. Newton): If $ABCD$ is a quadrilateral such that its sides AB , BC , CD and DA are tangent at M , N , P , and Q , respectively to a circle Γ , then AC , BD , MP and NQ are concurrent.

Proof: Let $AB \cap CD = \{R\}$, $AD \cap BC = \{S\}$, $AC \cap BD = \{I\}$ – see Figure 6. Then the polar of R is SI and also the polar of R is PM . From the uniqueness of the polar of a point with respect to a circle, we get that the points S , P , I , M are collinear (1). Similarly, the polar of S is RI and the polar of S is QN , resulting in the points R , Q , I , N being collinear (2). From relations (1) and (2), we find that the lines AC , BD , PM and NQ are concurrent.

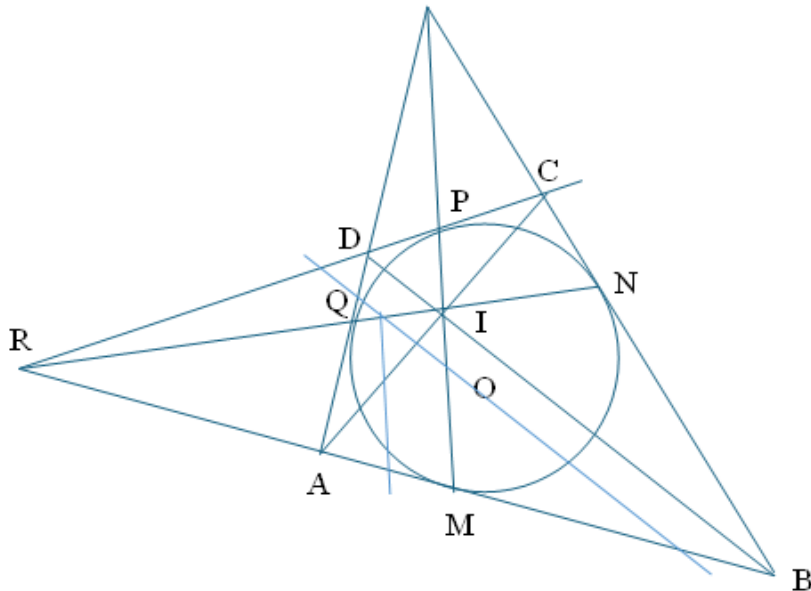


Figure 6. Illustration demonstrating Theorem 4 by I. Newton.

Theorem 5: Let $ABCD$ be a quadrilateral inscribed in the circle $\Gamma(O, r)$, with $\{I\} = AC \cap BD$. Let $\{R\} = AB \cap CD$, $\{S\} = AD \cap BC$, Q be the intersection of the tangents drawn at A and C to the circle Γ , and P be the intersection of the tangents drawn at B and D to the circle Γ . Then the points P, Q, R, S are collinear.

Proof: The polar of point P is BD . The polar of point Q is AC . The polar of point S is line RI , and the polar of point R is SI . Since the polars AC, BD, RI , and SI are concurrent lines at point I , then, according to *Theorem 2*, it follows that their poles, that is, the points Q, P, S and R , are collinear points.

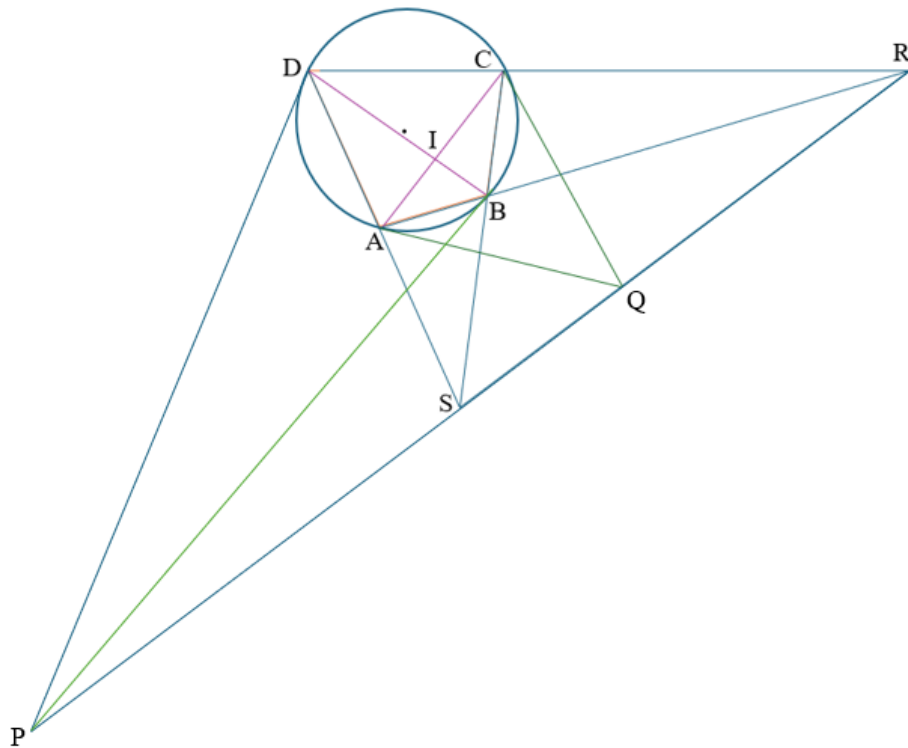


Figure 7. Illustration demonstrating the collinearity of points P, Q, R , and S in quadrilateral $ABCD$ inscribed in circle $\Gamma(O, r)$.

4 | Conclusion

The concepts of pole and polar with respect to a circle emerge as powerful tools in the study of geometry, offering elegant solutions to a diverse array of problems. Through our exploration of various applications, from determining the orthogonality of circles to establishing the concurrence of lines in inscribed quadrilaterals, we have witnessed the versatility and beauty of these concepts. Whether applied to classic theorems or contemporary problems, pole and polar relationships illuminate the inherent structure and symmetry within geometric configurations. Moreover, their connections to harmonic conjugates, inversive transformations, and projective geometry underscore their significance beyond the realm of pure mathematics, extending their utility to fields such as physics, engineering, and computer science. As we conclude this endeavor, we are reminded of the enduring relevance and timeless elegance of these fundamental concepts, inviting further inquiry and exploration into the depths of geometric theory and its myriad applications.

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Author Contribution

All authors contributed equally to this work.

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Data Availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare that there is no conflict of interest in the research.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

References

- [1] Pedoe, Daniel. Circles: a mathematical view. Cambridge University Press, 1995.
- [2] Gordon, R. G., Cox, A., & O'Hare, S. (1984). Paleomagnetic Euler poles and the apparent polar wander and absolute motion of North America since the Carboniferous. *Tectonics*, 3(5), 499-537.
- [3] Gray, J., & Gray, J. (2010). Poncelet (and Pole and Polar). *Worlds Out of Nothing: A Course in the History of Geometry in the 19th Century*, 11-24.
- [4] Ueda, K. (2000). Polar curves and surfaces. In *The Mathematics of Surfaces IX: Proceedings of the Ninth IMA Conference on the Mathematics of Surfaces* (pp. 372-388). Springer London.
- [5] R.A. Johnson, *Advanced Euclidian Geometry*. New York: Dover Publication, 2007
- [6] F. Smarandache, I. Patrascu, *The Geometry of Homological Triangles*. Columbus: Educational Publisher, 2012
- [7] I. Pătrașcu, F. Smarandache, *Geometria triunghiurilor ortologice (The Geometry of Orthologic Triangles)*. Craiova: Sitech, 2020

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