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A Unified Graph-Based Framework for Modeling Legal Citations: From Graphs to HyperGraphs and SuperHyperGraphs

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Abstract

Hypergraphs extend traditional graphs by allowing each hyperedge to join any number of vertices simultaneously, while superhypergraphs further enrich this idea through iterated powerset layers that capture hierarchical and self-referential connections. Graph theory has also proven valuable in legal scholarship: *Legal Citation Graphs* model precedential relationships by representing legal documents as nodes and citations as directed edges.

In this paper, we introduce two new formalisms—Legal Citation Hypergraphs and Legal Citation n -SuperHyperGraphs—that generalize citation networks to higher-order and multi-layered structures. We provide precise mathematical definitions, establish foundational properties, and discuss how these models illuminate complex citation patterns in legal systems.

Keywords: Superhypergraph, Hypergraph, Legal Citation Graph, Legal Citation HyperGraph

1 | Preliminaries

This section introduces the fundamental concepts and definitions that underpin the discussions in this paper. Throughout, all sets are assumed to be finite.

1.1 | Power Set and n -th Power Set

We provide the definitions of the power set and the n -th power set below. These concepts are used in the definition of Superhypergraphs.

Definition 1 (Universal Set). Let U be a *universal set* containing every object under consideration. All other sets in this work are assumed to satisfy $S \subseteq U$.



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Definition 2 (Base Set). A *base set* S is any subset of U from which more complex constructions originate:

$$S \subseteq U.$$

All subsequent constructions such as powersets or higher-order powersets begin with elements of S .

Definition 3 (Power Set). The *power set* of S , denoted $\mathcal{P}(S)$, is the set of all subsets of S :

$$\mathcal{P}(S) = \{ A \mid A \subseteq S \}.$$

Definition 4 (n -th Power Set). [1, 2] Define recursively the n -th *power set* of S by

$$\mathcal{P}^{(1)}(S) = \mathcal{P}(S), \quad \mathcal{P}^{(n+1)}(S) = \mathcal{P}(\mathcal{P}^{(n)}(S)), \quad n \geq 1.$$

Similarly, the n -th *nonempty power set* of S is

$$\mathcal{P}^{*(1)}(S) = \mathcal{P}(S) \setminus \{\emptyset\}, \quad \mathcal{P}^{*(n+1)}(S) = \mathcal{P}(\mathcal{P}^{*(n)}(S)) \setminus \{\emptyset\}.$$

Example 5 (Hierarchical Travel-Planning with Higher-Order Power Sets). **Base level** ($n = 0$). Let the base set of *single activities* be

$$S = \{\text{Museum, Beach, Hiking}\}.$$

First power set ($n = 1$). The ordinary power set

$$\mathcal{P}^{(1)}(S) = \mathcal{P}(S)$$

lists every *possible set of activities* that can be scheduled in one day, e.g. {Museum, Beach} (city-and-coast day) or {Hiking} (outdoor day).

Second power set ($n = 2$). An element of

$$\mathcal{P}^{(2)}(S) = \mathcal{P}(\mathcal{P}^{(1)}(S))$$

is a *set of day-plans*. For instance

$$D = \left\{ \{\text{Museum}\}, \{\text{Beach, Hiking}\} \right\}$$

describes a *two-day itinerary*: day 1 at the museum, day 2 combining beach and hiking.

Third power set ($n = 3$). An element of

$$\mathcal{P}^{(3)}(S) = \mathcal{P}(\mathcal{P}^{(2)}(S))$$

is a *collection of itineraries*—for example the packages offered by a travel agency:

$$\mathcal{I} = \left\{ \underbrace{\{\{\text{Museum}\}, \{\text{Beach, Hiking}\}\}}_{\text{“Culture \& Nature” package}}, \right. \\ \left. \underbrace{\{\{\text{Beach}\}, \{\text{Beach}\}, \{\text{Hiking}\}\}}_{\text{“Two Coasts + Hike” package}} \right\}.$$

Interpretation. $\mathcal{P}^{(1)}(S)$ models **daily activity bundles**; $\mathcal{P}^{(2)}(S)$ models **multi-day itineraries**; $\mathcal{P}^{(3)}(S)$ models **catalogues of itineraries**. Thus the n -th power set naturally captures increasingly higher organisational layers in real-world planning.

1.2 | SuperHyperGraph

In classical graph theory, a hypergraph extends the idea of a conventional graph by permitting edges—called hyperedges—to join more than two vertices. This broader framework enables the modeling of more intricate relationships between elements, thereby enhancing its utility in various fields [3, 4]. A *SuperHyperGraph* is an advanced extension of the hypergraph concept, integrating recursive powerset structures into the classical model. This concept has been recently introduced and extensively studied in the literature [5, 6, 7, 8].

Definition 6 (Hypergraph). [9, 3] A *hypergraph* $H = (V(H), E(H))$ consists of:

- A nonempty set $V(H)$ of vertices.
- A set $E(H)$ of hyperedges, where each hyperedge is a nonempty subset of $V(H)$, thereby allowing connections among multiple vertices.

Unlike standard graphs, hypergraphs are well-suited to represent higher-order relationships. In this paper, we restrict ourselves to the case where both $V(H)$ and $E(H)$ are finite.

Example 7 (Real-life hypergraph: recipes as ingredient sets). Let the vertex set (ingredients) be

$$V(H) = \{\text{Tomato, Basil, Mozzarella, Pasta, OliveOil}\}.$$

Define three hyperedges (dishes) as ingredient subsets:

$$e_{\text{Caprese}} = \{\text{Tomato, Basil, Mozzarella, OliveOil}\},$$

$$e_{\text{Pomodoro}} = \{\text{Pasta, Tomato, OliveOil, Basil}\},$$

$$e_{\text{PestoPasta}} = \{\text{Pasta, Basil, OliveOil}\}.$$

Then the finite hypergraph

$$H = (V(H), E(H)), \quad E(H) = \{e_{\text{Caprese}}, e_{\text{Pomodoro}}, e_{\text{PestoPasta}}\},$$

models higher-order relationships: each dish connects multiple ingredients simultaneously ($e \in E(H) \Rightarrow \emptyset \neq e \subseteq V(H)$).

Definition 8 (n -SuperHyperGraph). [10, 11]

Let V_0 be a finite nonempty set of basic vertices. For each integer $k \geq 0$, define the iterated powerset by

$$\mathcal{P}^0(V_0) = V_0, \quad \mathcal{P}^{k+1}(V_0) = \mathcal{P}(\mathcal{P}^k(V_0)),$$

where $\mathcal{P}(X)$ denotes the usual powerset of X .

An n -*SuperHyperGraph* is an ordered pair

$$\text{SHG}^{(n)} = (V, E),$$

such that

$$V \subseteq \mathcal{P}^n(V_0), \quad E \subseteq \mathcal{P}(V).$$

The elements of V are called n -*supervertices*, and the elements of E are called n -*superedges*.

Example 9 (Company–Programme collaboration as a 2-SuperHyperGraph). We instantiate a 2-SuperHyperGraph in three layers—individuals ($n = 0$), task-forces ($n = 1$), and multi-task-force programmes ($n = 2$)—and then form hyperedges between programmes.

Base level ($n = 0$). Let the ground vertex set be

$$V_0 = \{\text{Masahiro, Yutaka, Yoshio, Dave, Eve}\}.$$

First powerset level ($n = 1$). Nonempty subsets of V_0 represent *task-forces*:

$$T_{\text{UI}} = \{\text{Masahiro, Yutaka}\}, \quad T_{\text{AI}} = \{\text{Yoshio, Dave}\}, \quad T_{\text{Cloud}} = \{\text{Eve}\}.$$

Thus $T_{\text{UI}}, T_{\text{AI}}, T_{\text{Cloud}} \in \mathcal{P}(V_0) \setminus \{\emptyset\}$.

Second powerset level ($n = 2$). Nonempty subsets of the task-force family represent *programmes*:

$$P_{\text{Mobile}} = \{ T_{\text{UI}}, T_{\text{Cloud}} \}, \quad P_{\text{Analytics}} = \{ T_{\text{AI}}, T_{\text{Cloud}} \}.$$

Consequently $P_{\text{Mobile}}, P_{\text{Analytics}} \in \mathcal{P}^2(V_0)$.

2-supervertices. Set

$$V^{(2)} = \{ P_{\text{Mobile}}, P_{\text{Analytics}} \} \subseteq \mathcal{P}^2(V_0),$$

whose elements are the 2-supervertices (programmes).

Superedges (programme collaborations). Hyperedges are nonempty subsets of $V^{(2)}$:

$$e_{\text{PI}} = \{ P_{\text{Mobile}}, P_{\text{Analytics}} \} \quad (\text{“Product Innovation” joint initiative}),$$

$$e_{\text{SDI}} = \{ P_{\text{Analytics}} \} \quad (\text{stand-alone data initiative}).$$

Define

$$E^{(2)} = \{ e_{\text{PI}}, e_{\text{SDI}} \} \subseteq \mathcal{P}(V^{(2)}) \setminus \{\emptyset\}.$$

2-SuperHyperGraph. The ordered pair

$$\text{SHG}^{(2)} = (V^{(2)}, E^{(2)})$$

is a 2-SuperHyperGraph: by construction $V^{(2)} \subseteq \mathcal{P}^2(V_0)$ and $E^{(2)} \subseteq \mathcal{P}(V^{(2)}) \setminus \{\emptyset\}$. This model cleanly captures a three-tier corporate structure: individual engineers ($n = 0$), task-forces ($n = 1$), and multi-task-force programmes ($n = 2$), with programme-level collaborations encoded as hyperedges.

Example 10 (Real-life 3-SuperHyperGraph: university structure). Write $\mathcal{P}^0(V_0) = V_0$ and $\mathcal{P}^{k+1}(V_0) = \mathcal{P}(\mathcal{P}^k(V_0))$, where $\mathcal{P}$ is the powerset.

Base level ($n = 0$; students):

$$V_0 = \{ \text{Alice}, \text{Bob}, \text{Carol}, \text{Dan}, \text{Emma} \}.$$

Level 1 ($\mathcal{P}^1(V_0)$; courses as student groups):

$$C_{\text{Alg}} = \{ \text{Alice}, \text{Bob} \}, \quad C_{\text{DS}} = \{ \text{Bob}, \text{Carol} \}, \quad C_{\text{Bio}} = \{ \text{Dan}, \text{Emma} \}.$$

Level 2 ($\mathcal{P}^2(V_0)$; majors as sets of courses):

$$M_{\text{CS}} = \{ C_{\text{Alg}}, C_{\text{DS}} \}, \quad M_{\text{Bio}} = \{ C_{\text{Bio}} \}.$$

Level 3 ($\mathcal{P}^3(V_0)$; colleges as sets of majors):

$$\text{Col}_{\text{Eng}} = \{ M_{\text{CS}} \}, \quad \text{Col}_{\text{Sci}} = \{ M_{\text{Bio}} \}.$$

3-supervertices and superedges:

$$V^{(3)} = \{ \text{Col}_{\text{Eng}}, \text{Col}_{\text{Sci}} \} \subseteq \mathcal{P}^3(V_0),$$

$$e_{\text{Inter}} = \{ \text{Col}_{\text{Eng}}, \text{Col}_{\text{Sci}} \} \quad (\text{interdisciplinary program}), \quad e_{\text{Accr}} = \{ \text{Col}_{\text{Sci}} \} \quad (\text{standalone accreditation activity}),$$

$$E^{(3)} = \{ e_{\text{Inter}}, e_{\text{Accr}} \} \subseteq \mathcal{P}(V^{(3)}) \setminus \{\emptyset\}.$$

Thus

$$\text{SHG}^{(3)} = (V^{(3)}, E^{(3)})$$

is a 3-SuperHyperGraph: $V^{(3)} \subseteq \mathcal{P}^3(V_0)$ and each $e \in E^{(3)}$ is a nonempty subset of $V^{(3)}$. It captures three real-life layers (students $\rightarrow$ courses $\rightarrow$ majors $\rightarrow$ colleges) with college-level collaborations encoded as hyperedges.

1.3 | Legal Citation Graphs

Legal citation is the practice of referencing legal documents, such as statutes or cases, to support arguments or decisions (cf.[12, 13, 14]). Legal Citation Graphs represent legal documents as nodes and citations between them as directed edges, modeling legal precedent structure [15, 16].

Definition 11 (Legal Citation Graph). A *Legal Citation Graph* is a directed, labeled graph

$$G = (V, E, \ell),$$

where

- V is a finite set of *provisions*, each element $v \in V$ representing a distinct unit of legal text (e.g. a section, article, or clause).
- $E \subseteq V \times V$ is the set of *citation edges*, where $(u, v) \in E$ if and only if provision u contains a citation to provision v .
- $\ell : E \rightarrow L$ is an *edge-labeling function* assigning to each citation edge (u, v) a semantic label $\ell(u, v)$ drawn from a predefined label set L (e.g. Definition, Exception, Authority, etc.).

We may also consider the *weighted* variant

$$G_w = (V, E, w, \ell),$$

where $w : E \rightarrow \mathbb{R}_{>0}$ assigns to each edge a weight reflecting, for example, citation frequency or strength of dependency.

Example 12 (Legal Citation Graph: a small statutory/case network). Let the label set be

$$L = \{\text{Definition, Authority, Exception, Implementation}\}.$$

Consider the finite provision set

$$V = \{\text{DPA_Sec2, DPA_Sec12, DPA_Sec18, Reg_R5, Case_2024_HC}\},$$

interpreted as: DPA_Sec2 (Definitions), DPA_Sec12 (Lawful Processing), DPA_Sec18 (Exemptions), Reg_R5 (Commission Regulation Art. 5), Case_2024_HC (High Court judgment, 2024).

Define directed citation edges $E \subseteq V \times V$ by

$$E = \{(\text{DPA_Sec12, DPA_Sec2}), (\text{DPA_Sec12, DPA_Sec18}), (\text{Reg_R5, DPA_Sec12}), \\ (\text{Case_2024_HC, DPA_Sec12}), (\text{Case_2024_HC, DPA_Sec18}), (\text{Case_2024_HC, DPA_Sec2})\}.$$

Assign semantic labels $\ell : E \rightarrow L$ by

$$\begin{aligned} \ell(\text{DPA_Sec12, DPA_Sec2}) &= \text{Definition}, \\ \ell(\text{DPA_Sec12, DPA_Sec18}) &= \text{Exception}, \\ \ell(\text{Reg_R5, DPA_Sec12}) &= \text{Implementation}, \\ \ell(\text{Case_2024_HC, DPA_Sec12}) &= \text{Authority}, \\ \ell(\text{Case_2024_HC, DPA_Sec18}) &= \text{Authority}, \\ \ell(\text{Case_2024_HC, DPA_Sec2}) &= \text{Definition}. \end{aligned}$$

Optionally, a weighted variant $G_w = (V, E, w, \ell)$ can encode citation strength/frequency via

$$\begin{aligned} w(\text{DPA_Sec12, DPA_Sec2}) &= 3.0, \\ w(\text{DPA_Sec12, DPA_Sec18}) &= 1.0, \\ w(\text{Reg_R5, DPA_Sec12}) &= 2.0, \\ w(\text{Case_2024_HC, DPA_Sec12}) &= 2.5, \\ w(\text{Case_2024_HC, DPA_Sec18}) &= 1.5, \\ w(\text{Case_2024_HC, DPA_Sec2}) &= 1.0. \end{aligned}$$

Then $G = (V, E, \ell)$ is a Legal Citation Graph in the sense of the definition: each $(u, v) \in E$ records that provision u cites provision v , with $\ell(u, v)$ capturing the citation's semantics (definition reference, authority, exception, or implementation).

2 | Main Results in This Paper

The results presented in this paper are explained below.

2.1 | Legal Citation HyperGraphs

We present the definition, concrete examples, and theorems of Legal Citation Hypergraph.

Definition 13 (Legal Citation Hypergraph). A *Legal Citation Hypergraph* is a quadruple

$$H = (V, E, s, \ell),$$

where

- V is a finite set of *provisions*, each $v \in V$ representing a distinct unit of legal text (e.g. section, article, or clause).
- $E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$ is a finite family of *hyperedges*, each $e \in E$ representing one citation event.
- $s : E \rightarrow V$ is a *source function* such that for every $e \in E$, the *citing provision* $s(e)$ satisfies $s(e) \in e$. The *cited provisions* of e are $e \setminus \{s(e)\}$.
- $\ell : E \rightarrow L$ is an *edge-labeling function* assigning to each hyperedge a semantic label from a set L of citation types (e.g. “Definition,” “Exception,” “Authority”).

Example 14 (Judicial Precedent Citation in *Obergefell v. Hodges*). *Obergefell v. Hodges* is the 2015 U.S. Supreme Court decision that legalized same-sex marriage nationwide under the Constitution (cf.[17]). Consider the following set of legal provisions:

$$V = \{v_O, v_B, v_R\},$$

where

- v_O denotes the U.S. Supreme Court decision *Obergefell v. Hodges* (2015),
- v_B denotes *Brown v. Board of Education* (1954),
- v_R denotes *Roe v. Wade* (1973).

Define the hyperedge family

$$E = \{e\}, \quad e = \{v_O, v_B, v_R\}.$$

Then the source and labeling functions are given by

$$s(e) = v_O, \quad \ell(e) = \text{“Authority”}.$$

Here, the single hyperedge e models the citation event in which the Supreme Court in *Obergefell v. Hodges* cited *Brown v. Board of Education* and *Roe v. Wade* as controlling precedents for its constitutional analysis.

Example 15 (Massachusetts v. EPA Citation as a Hypergraph). *Massachusetts v. EPA* is a 2007 U.S. Supreme Court case affirming the EPA's authority to regulate greenhouse gases under the Clean Air Act (cf.[18, 19]). Consider the following provisions:

$$V = \{v_M, v_S, v_P\},$$

where

- v_M denotes the U.S. Supreme Court decision *Massachusetts v. EPA* (2007),
- v_S denotes Section 202(a)(1) of the Clean Air Act,
- v_P denotes the Supreme Court decision *Chevron U.S.A., Inc. v. NRDC* (1984).

Define the hyperedge family

$$E = \{ e \}, \quad e = \{ v_M, v_S, v_P \}.$$

Then the source and labeling functions are

$$s(e) = v_M, \quad \ell(e) = \text{“Authority”}.$$

- The hyperedge e models the citation event in which *Massachusetts v. EPA* cites both the statutory provision (Clean Air Act §202(a)(1)) and the judicial precedent (*Chevron U.S.A., Inc. v. NRDC*).
- Since $s(e) \in e$, v_M is correctly identified as the citing provision.
- The label $\ell(e) = \text{“Authority”}$ indicates that these citations serve as binding authority for the Court’s decision.

Theorem 16 (Hypergraph Property). *Let $H = (V, E, s, \ell)$ be a Legal Citation Hypergraph. Then the pair (V, E) satisfies the standard definition of a hypergraph.*

Proof: By definition, V is a nonempty finite set and $E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$. Hence each $e \in E$ is a nonempty subset of V . These are exactly the requirements for (V, E) to be a hypergraph as in [9, 3]. $\square$

Theorem 17 (Generalization of Legal Citation Graph). *Let $H = (V, E, s, \ell)$ be a Legal Citation Hypergraph. Define a directed graph*

$$G = (V, E', \ell'),$$

where

$$E' = \{ (s(e), v) \mid e \in E, v \in e \setminus \{s(e)\} \}, \quad \ell'(s(e), v) = \ell(e).$$

Then G is precisely a Legal Citation Graph (cf. Definition 1), and hence H generalizes any such graph.

Proof: First, $E' \subseteq V \times V$ by construction. For each $(u, v) \in E'$, there exists $e \in E$ with $u = s(e)$ and $v \in e \setminus \{u\}$; thus provision u cites provision v . The labeling ℓ' inherits the semantic type from the underlying hyperedge e . These conditions exactly match those of a Legal Citation Graph, where directed edges represent individual citations and carry the same labels. Therefore, the 2-section (directed) graph G of H recovers the original Legal Citation Graph, proving that H indeed generalizes it. $\square$

Theorem 18 (Two-Section Representation). *Let $H = (V, E, s, \ell)$ be a Legal Citation Hypergraph. Define the directed graph*

$$G = (V, E', \ell'), \quad E' = \{ (s(e), v) \mid e \in E, v \in e \setminus \{s(e)\} \}, \quad \ell'(s(e), v) = \ell(e).$$

Then G is a Legal Citation Graph.

Proof: By construction, $E' \subseteq V \times V$. For each $(u, v) \in E'$ there exists $e \in E$ with $u = s(e)$ and $v \in e \setminus \{u\}$, so provision u cites provision v . The labeling ℓ' inherits the semantic label from the hyperedge e . These properties exactly match the definition of a Legal Citation Graph. $\square$

Theorem 19 (Connectivity Equivalence). *Let $H = (V, E, s, \ell)$ be a Legal Citation Hypergraph, and let G be its undirected 2-section graph with edge set*

$$\{ \{u, v\} \mid \exists e \in E, u, v \in e \}.$$

Then H is connected (as a hypergraph) if and only if G is connected (as a graph).

Proof: Suppose H is connected: for any $u, v \in V$ there exists a sequence

$$u = v_0, e_1, v_1, e_2, v_2, \dots, e_k, v_k = v \quad \text{with} \quad v_{i-1}, v_i \in e_i.$$

Then in G , each $\{v_{i-1}, v_i\}$ is an edge, so $(v_0, v_1, \dots, v_k)$ is a path in G .

Conversely, if G is connected, for any $u, v \in V$ there is a path $\{u = w_0, w_1\}, \{w_1, w_2\}, \dots, \{w_{m-1}, w_m = v\}$ in G . By definition of G , each $\{w_{i-1}, w_i\}$ lies in some hyperedge e_i of H . Thus

$$u = w_0, e_1, w_1, e_2, w_2, \dots, e_m, w_m = v$$

is a hypergraph walk in H , proving H is connected. $\square$

Theorem 20 (Citation Relation Induces a Partial Order). *Let $H = (V, E, s, \ell)$ be a Legal Citation Hypergraph, and let*

$$R = \{(u, v) \in V \times V \mid \exists e \in E : s(e) = u, v \in e \setminus \{u\}\}.$$

Assume the directed graph (V, R) has no directed cycles. Define a relation $\leq$ on V by

$$u \leq v \iff u = v \text{ or there is a finite sequence } u = x_0 R x_1 R \dots R x_k = v.$$

Then $\leq$ is a partial order on V .

Proof: We verify the three properties:

Reflexivity: For any $u \in V$, $u = u$ implies $u \leq u$.

Antisymmetry: Suppose $u \leq v$ and $v \leq u$. If $u \neq v$, then there would be directed paths $uR \dots Rv$ and $vR \dots Ru$, forming a directed cycle in (V, R) , contradicting the acyclicity assumption. Hence $u = v$.

Transitivity: If $u \leq v$ via $uR \dots Rv$ and $v \leq w$ via $vR \dots Rw$, then concatenating these yields a path $uR \dots Rw$, so $u \leq w$.

Thus $\leq$ is reflexive, antisymmetric, and transitive, and therefore a partial order. $\square$

2.2 | Legal Citation n -SuperHyperGraphs

We provide a concise, mathematically sound definition of Legal Citation n -SuperHyperGraphs, together with concrete examples and basic properties.

Definition 21 (Legal Citation n -SuperHyperGraph). Let V_0 be a finite, nonempty set of legal provisions (statutes, regulations, cases, guidance, etc.). For each integer $k \geq 0$, define the iterated powerset

$$\mathcal{P}^0(V_0) = V_0, \quad \mathcal{P}^{k+1}(V_0) = \mathcal{P}(\mathcal{P}^k(V_0)),$$

where $\mathcal{P}(X)$ denotes the powerset of X .

A Legal Citation n -SuperHyperGraph is a triple

$$\text{LCSHG}^{(n)} = (V^{(n)}, E^{(n)}, \ell^{(n)}),$$

with the following data:

- $V^{(n)} \subseteq \mathcal{P}^n(V_0)$ is the finite set of n -supervertices (nested provision groups).
- $E^{(n)} \subseteq V^{(n)} \times (\mathcal{P}(V^{(n)}) \setminus \{\emptyset\})$ is the finite set of *oriented n -superedges*. Each edge is an ordered pair
$$e = (u, T), \quad u \in V^{(n)}, \quad \emptyset \neq T \subseteq V^{(n)},$$
read as “the citing supervertex u cites all targets in T ”. By convention we exclude self-citation and assume $u \notin T$ (this can be relaxed if desired).
- $\ell^{(n)} : E^{(n)} \rightarrow L$ is an *edge-labeling* into a fixed finite label set L (e.g. **Authority**, **Exception**, **InterpretiveGuidance**).

For $e = (u, T) \in E^{(n)}$, write $\text{src}(e) = u$ and $\text{tgt}(e) = T$.

Example 22 (EPA report citing multiple frameworks (level $n = 2$)). Let

$$V_0 = \{v_{\text{EPA}}, v_{\text{CAA108}}, v_{\text{CWA201}}, v_{\text{OMB_A4}}\}.$$

Define the 2-supervertices

$$X_{\text{EPA}} = \{\{v_{\text{EPA}}\}\}, \quad X_{\text{ENV}} = \{\{v_{\text{CAA108}}, v_{\text{CWA201}}\}\}, \quad X_{\text{OMB}} = \{\{v_{\text{OMB_A4}}\}\},$$

and set

$$V^{(2)} = \{X_{\text{EPA}}, X_{\text{ENV}}, X_{\text{OMB}}\} \subseteq \mathcal{P}^2(V_0).$$

Introduce one oriented 2-superedge

$$e = (X_{\text{EPA}}, \{X_{\text{ENV}}, X_{\text{OMB}}\}) \in V^{(2)} \times (\mathcal{P}(V^{(2)}) \setminus \{\emptyset\}),$$

meaning the EPA report cites both the grouped environmental statutes and OMB Circular A-4. Let $\ell^{(2)}(e) = \text{RegulatoryCites}$. Then $\text{LCSHG}^{(2)} = (V^{(2)}, \{e\}, \ell^{(2)})$.

Example 23 (Composite citations in *West Virginia v. EPA* (level $n = 2$)). Let

$$V_0 = \{v_{\text{WV}}, v_{\text{CAA741}}, v_{\text{Chevron}}, v_{\text{HLR_MQD}}\},$$

where v_{WV} denotes the 2022 U.S. Supreme Court decision *West Virginia v. EPA*, v_{CAA741} a CAA section, v_{Chevron} the 1984 *Chevron* precedent, and $v_{\text{HLR_MQD}}$ a law-review article on the Major Questions Doctrine. Define

$$X_{\text{WV}} = \{\{v_{\text{WV}}\}\}, \quad X_{\text{Auth}} = \{\{v_{\text{CAA741}}, v_{\text{Chevron}}\}\}, \quad X_{\text{Comm}} = \{\{v_{\text{HLR_MQD}}\}\},$$

and take $V^{(2)} = \{X_{\text{WV}}, X_{\text{Auth}}, X_{\text{Comm}}\}$. Let

$$e = (X_{\text{WV}}, \{X_{\text{Auth}}, X_{\text{Comm}}\}), \quad \ell^{(2)}(e) = \text{CompositeAuthority}.$$

Then $\text{LCSHG}^{(2)} = (V^{(2)}, \{e\}, \ell^{(2)})$ encodes that the decision simultaneously cites statutory/judicial authority and academic commentary.

Example 24 (Amendment citing prior sections (level $n = 2$)). Let

$$V_0 = \{v_{S1}, v_{S2}, v_{S3}, v_{\text{AmendA}}\}.$$

Define

$$X_{12} = \{\{v_{S1}, v_{S2}\}\}, \quad X_3 = \{\{v_{S3}\}\}, \quad X_A = \{\{v_{\text{AmendA}}\}\},$$

set $V^{(2)} = \{X_{12}, X_3, X_A\}$, and add the edge

$$e = (X_A, \{X_{12}, X_3\}), \quad \ell^{(2)}(e) = \text{AmendmentCites}.$$

Thus $\text{LCSHG}^{(2)} = (V^{(2)}, \{e\}, \ell^{(2)})$ models an amendment that cites two composite provision groups.

Theorem 25 (One-mode projection to a citation digraph). *Given $\text{LCSHG}^{(n)} = (V^{(n)}, E^{(n)}, \ell^{(n)})$, define the directed graph*

$$\Gamma(\text{LCSHG}^{(n)}) = (V^{(n)}, A), \quad A = \{(\text{src}(e), v) : e \in E^{(n)}, v \in \text{tgt}(e)\}.$$

Then for any $u, v \in V^{(n)}$, there exists a directed path $u \rightsquigarrow v$ in $\Gamma(\text{LCSHG}^{(n)})$ if and only if there exists a finite sequence of superedges

$$e_1, \dots, e_m \in E^{(n)} \quad \text{with} \quad \text{src}(e_1) = u, \quad v \in \text{tgt}(e_m), \quad \text{tgt}(e_i) \ni \text{src}(e_{i+1})$$

for all $i = 1, \dots, m-1$.

Proof: ($\Rightarrow$): A directed path $u = v_0 \rightarrow v_1 \rightarrow \dots \rightarrow v_m = v$ in Γ means that for each i there exists $e_i \in E^{(n)}$ with $\text{src}(e_i) = v_{i-1}$ and $v_i \in \text{tgt}(e_i)$. Setting these e_i yields the required sequence.

($\Leftarrow$): Given $e_1, \dots, e_m$ with $\text{src}(e_1) = u$, $v \in \text{tgt}(e_m)$, and $\text{tgt}(e_i) \ni \text{src}(e_{i+1})$, pick $v_i \in \text{tgt}(e_i)$ so that $v_i = \text{src}(e_{i+1})$ for $i < m$, and $v_m = v$. Then $(v_{i-1}, v_i) \in A$ for all i , giving a path $u \rightsquigarrow v$ in Γ . $\square$

Theorem 26 (Acyclicity and topological order). *If $\Gamma(\text{LCSHG}^{(n)})$ is acyclic, then there exists a topological ordering of $V^{(n)}$ consistent with all citations, i.e. whenever $e = (u, T) \in E^{(n)}$ and $v \in T$, the order places u strictly before v .*

Proof: This is the standard property of finite DAGs: acyclicity implies the existence of a linear extension (topological order) of the reachability partial order on $V^{(n)}$. Since every arc of Γ comes from some $e = (u, T)$ with $v \in T$, the order respects all citation directions. $\square$

Theorem 27 (Layer reduction (one-level projection)). *Fix $n \geq 1$. Let $\text{LCSHG}^{(n)} = (V^{(n)}, E^{(n)}, \ell^{(n)})$ be a Legal Citation n -SuperHyperGraph in the sense of Definition 21 (i.e. each $e \in E^{(n)}$ is an ordered pair $e = (u, T)$ with $u \in V^{(n)}$, $\emptyset \neq T \subseteq V^{(n)}$, interpreted as “ u cites every $v \in T$ ”). Define the $(n-1)$ -level vertex set by*

$$V^{(n-1)} := \bigcup_{U \in V^{(n)}} U \subseteq \mathcal{P}^{n-1}(V_0),$$

and the $(n-1)$ -level edge set by

$$E^{(n-1)} := \left\{ (x, T^\downarrow \setminus \{x\}) : (u, T) \in E^{(n)}, x \in u, T^\downarrow := \bigcup_{v \in T} v, T^\downarrow \setminus \{x\} \neq \emptyset \right\}.$$

Equip each new edge with the inherited label $\ell^{(n-1)}(x, T^\downarrow \setminus \{x\}) := \ell^{(n)}(u, T)$. Then

$$\text{LCSHG}^{(n-1)} := (V^{(n-1)}, E^{(n-1)}, \ell^{(n-1)})$$

is a Legal Citation $(n-1)$ -SuperHyperGraph.

Proof: (Vertices) For each $U \in V^{(n)}$ we have $U \subseteq \mathcal{P}^{n-1}(V_0)$ by definition of $V^{(n)} \subseteq \mathcal{P}^n(V_0)$; hence $\bigcup_{U \in V^{(n)}} U \subseteq \mathcal{P}^{n-1}(V_0)$, so $V^{(n-1)}$ is well-defined.

(Edges) Fix $e = (u, T) \in E^{(n)}$. Since $u \subseteq \mathcal{P}^{n-1}(V_0)$, every $x \in u$ lies in $\mathcal{P}^{n-1}(V_0)$, hence $x \in V^{(n-1)}$ by construction. Likewise $T^\downarrow = \bigcup_{v \in T} v \subseteq \bigcup_{U \in V^{(n)}} U = V^{(n-1)}$. Whenever $T^\downarrow \setminus \{x\} \neq \emptyset$, the ordered pair $(x, T^\downarrow \setminus \{x\})$ belongs to $V^{(n-1)} \times (\mathcal{P}(V^{(n-1)}) \setminus \{\emptyset\})$, so it is a valid $(n-1)$ -superedge. Collecting all such pairs over all $e \in E^{(n)}$ yields $E^{(n-1)}$.

(Labels) The assignment $\ell^{(n-1)}(x, T^\downarrow \setminus \{x\}) := \ell^{(n)}(u, T)$ is well-defined per source edge $e = (u, T)$. Thus $(V^{(n-1)}, E^{(n-1)}, \ell^{(n-1)})$ satisfies Definition 21 for level $n-1$. $\square$

Theorem 28 (Multi-section projection to citation digraphs). *Let $\text{LCSHG}^{(n)} = (V^{(n)}, E^{(n)}, \ell^{(n)})$ be given.*

(i) Level- n (one-mode) projection. *Define the labeled digraph*

$$\Gamma_n = (V^{(n)}, A_n, \lambda_n), \quad A_n := \{ (u, v) : \exists e = (u, T) \in E^{(n)} \text{ with } v \in T \}.$$

For any $(u, v) \in A_n$, choose $e = (u, T)$ with $v \in T$ and set $\lambda_n(u, v) := \ell^{(n)}(e)$. Then Γ_n recovers all pairwise citation relations among the n -supervertices.

(ii) Base-level projection (to V_0). For $k \geq 1$ and $S \in \mathcal{P}^k(V_0)$, write

$$\text{Flat}_k(S) := \underbrace{\bigcup \dots \bigcup}_{k \text{ times}} S \subseteq V_0$$

(the k -fold union). Define the labeled digraph

$$\Gamma_0 = (V_0, A_0, \lambda_0), \quad A_0 := \left\{ (a, b) \in V_0 \times V_0 : \exists e = (u, T) \in E^{(n)}, \exists X \in u, \exists v \in T, \exists Y \in v, \right. \\ \left. a \in \text{Flat}_{n-1}(X), b \in \text{Flat}_{n-1}(Y) \right\}.$$

For any $(a, b) \in A_0$, pick witnesses as above and set $\lambda_0(a, b) := \ell^{(n)}(e)$. Then Γ_0 recovers all pairwise base-provision citations induced by the higher-order edges of $\text{LCSHG}^{(n)}$.

Proof: (i) By construction, each $e = (u, T) \in E^{(n)}$ contributes directed arcs (u, v) to A_n for all $v \in T$; hence every pairwise relation encoded by e is present in Γ_n , with the same semantic label.

(ii) Fix $e = (u, T) \in E^{(n)}$. For any $X \in u$ and any $v \in T$ with $Y \in v$, every $a \in \text{Flat}_{n-1}(X)$ and $b \in \text{Flat}_{n-1}(Y)$ yields an arc $(a, b) \in A_0$, labeled by $\ell^{(n)}(e)$. Thus all binary base-level citations implied by the higher-order edge e are realized in Γ_0 . Conversely, any $(a, b) \in A_0$ has such witnesses, so it comes from some $e \in E^{(n)}$. $\square$

3 | Conclusion and Future Works

In this paper, we defined and examined the concepts of *Legal Citation Hypergraphs* and *Legal Citation n -SuperHyperGraphs*, exploring both their mathematical foundations and their relevance to real-world legal systems.

In future work, we aim to explore extensions of these models by incorporating uncertainty-handling frameworks such as the *fuzzy set* [20], *intuitionistic fuzzy set* [21, 22], *rough set* [23], and *neutrosophic set* [24, 25].

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Data Availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare that there is no conflict of interest in the research.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

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